

## Linear Instability Analysis of Taylor-Couette flow

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**Abstract** — This paper presents the linear stability analysis of the viscous, incompressible, shear driven flow between two concentric anticlockwise rotating cylinders. The main aim of this work is to study the effect of the Radius ratio,  $\eta$  and ratio of the Angular velocity,  $\Omega$  on the stability of the Taylor-Couette flow. The Basic state and 3D Perturbation form of the linearized governing equation, which represent the Stability of the Taylor-Couette flow is derived. Chebyshev Spectral Collocation method is used as a solution method and linear stability analysis is carried out to calculate the most unstable Eigen vale and corresponding Eigen function or Disturbances component. Critical Reynolds number is calculated for different value of the radius ratio ( $\eta=0.5$  &  $0.99$ ) and the angular velocity ratio. Here, Angular velocity ratio is taken as negative because the both cylinders are rotating in opposite directions and find out the Critical Reynolds number for various values of the negative angular velocity radius ratio. The results of this study indicates the Critical Reynolds number is decreased due to increase in the relative velocity between inner and outer cylinders. That will happened only in a case of the rotation speed of the inner cylinder is greater than the outer cylinders (i.e.  $\omega_1 > \omega_2$ ). While in a case of the  $\omega_1 < \omega_2$ , Critical Reynolds number is increased. That means reduction of the speed of the inner cylinder or increase the outer cylinder speed stabilize the flow. For  $\Omega = -0.9, -1$  &  $-1.1$  Critical Reynolds number are 234, 306 & 392 respectively.

**Keywords-** Stability; Taylor-Couette flow; Radius Ratio; Angular Velocity; Critical Reynolds number

### I. INTRODUCTION

Instability means only small disturbances is enough to change the flow configuration from one state to another state particular from laminar to turbulent. It is not necessary that conversion is only from laminar to turbulent but it may be from one laminar state to another complicated laminar state. In this paper we study the temporal stability of the Taylor-Couette flow that means Disturbances are measured over the period of time. Taylor-Couette flow means flow between two concentric cylinders. The Simple Visualization of the Taylor-Couette flow has shown in fig. 1. This type of the flow is first investigation by the G.I.Taylor and it is good aggregation with the experiment results [1]. G.I.Taylor was first successfully applied the linear classical theory for the solution of the differential equation which governing the flow stability and solve that equation with the help of Bessel function approximation which require the more computational effort. The stability of the Taylor-Couette flow in the presence of the radial temperature is study by Chandrasekhar [4-5]. If the one cylinder is fixed and another is rotating then it's called as Circular Couette flow [8]. In some cases inner cylinder is rotating as well as moving in axial direction while the outer cylinder is fixed [9]. Taylor-Couette flow is used in the Gaseous core nuclear reactor, polymer processing [10], MHD and solar distillation plant etc. Here, we use the Linear Classical theory to study the Instability of the Taylor-Couette flow. The Energy Gradient theory is also used to study the Instability of the Taylor-Couette flow [2-3]. The Non-linear theory is also used to analysis the T-C flow but it is complicated and time consuming methods. The stability of the Taylor-Couette flow corresponding to the non-axis symmetric modes of the disturbances are derived by Krueger [7].

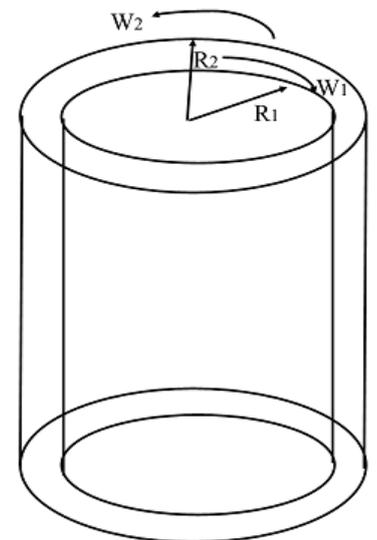


Fig. 1: Schematic Diagram of

#### Taylor-Couette flow.

The algorithm or Matlab code which is used to find the most unstable Eigen value & corresponding Eigen function is validate with the help of the transient growth in Taylor-Couette flow paper [6].

In this paper all distances are to be normalized by the gap width,  $d = (R_2 - R_1)/2$ , velocities are to be normalized by  $\omega_1 R_1$ . Our aim is to be find the effect of the Radius ratio and the Speed ratio on to the Stability of the Taylor-Couette flow and Compute the Critical Reynolds number.

The formation of this paper is organized as follow. The problem formulation are presented in sec. II. Mean flow or basic state solution are provided in sec. III. Linear Stability analysis is carried out in sec. IV. The results and Discussion are presented in sec. V.

## II. PROBLEM FORMULATION

The 3D Momentum and Continuity equation is dimensionless or normalized by the following terms,

$$t^* = \frac{td}{U}, r^* = \frac{r}{d}, p^* = \frac{p}{\rho U^2}, u^* = \frac{u}{U}, \Omega = \frac{\omega_2}{\omega_1} \quad (1)$$

Where, \* represent the Dimensionless quantity, V is the Mean velocity,  $d = (R_2 - R_1)/2$  is the Gap between two cylinders,  $R_1$  &  $R_2$  is the Radius of the inner and outer cylinder respectively,  $\omega_1$  &  $\omega_2$  is the angular velocity of the inner and outer cylinder respectively.

The Dimensionless form of the Governing Equation after dropping the \* becomes,

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0 \quad (2)$$

$$\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_r}{\partial \theta} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\theta}{r} = -\frac{\partial p}{\partial r} + \frac{1}{Re} \left\{ \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} - \frac{u_r}{r^2} \right\} \quad (3)$$

$$\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_\theta}{\partial \theta} + u_z \frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r} = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \frac{1}{Re} \left\{ \frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2} \right\} \quad (4)$$

$$\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_\theta}{\partial \theta} + u_z \frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r} = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \frac{1}{Re} \left\{ \frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2} \right\} \quad (5)$$

Where,

$Re = \rho U d / \mu$  is the Reynolds number based on the Gap width.  $u_r$ ,  $u_\theta$  &  $u_z$  are radial, tangential and axial component of the velocity.

## III. BASIC STATE

The basic state is derived by considering the flow is Steady, Parallel, Fully developed as well as function of the radial distance only. So, Dimensional Governing Equation reduced to,

$$\frac{\partial}{\partial r} \left( \frac{\partial \bar{u}_\theta}{\partial r} + \frac{\bar{u}_\theta}{r} \right) = 0 \quad \& \quad \frac{\partial \bar{p}}{\partial r} = \frac{\bar{u}_\theta^2}{r} \quad (6)$$

Where, '-'; represent the mean quantity.

Integration of the equation (6) twice with respect to 'r' and we get,

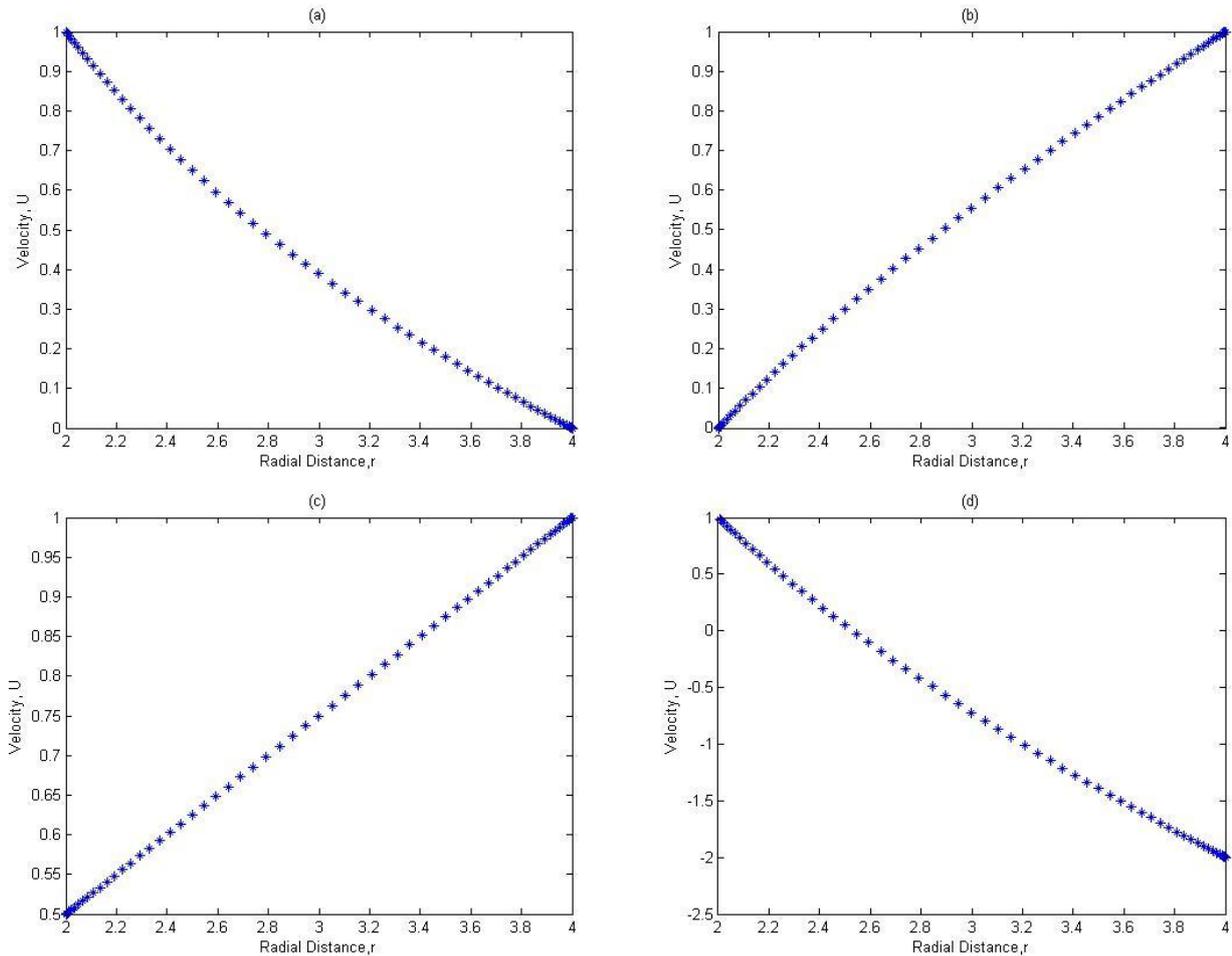
$$\bar{u}_\theta = \frac{Ar}{2} + \frac{B}{r} \quad (7)$$

Where, A & B are the constant of the integration and it is evaluated by the following Boundary Conditions:

$$\text{At } r = R_1, \bar{u}_\theta = \omega_1 R_1 \quad \& \quad r = R_2, \bar{u}_\theta = \omega_2 R_2$$

$$\text{So, } A = \frac{2(\omega_1 R_1^2 - \omega_2 R_2^2)}{R_1^2 - R_2^2} \quad \& \quad B = (2\omega_1 R_1^2 - AR_1^2) \quad (8)$$

Equation (7) is normalized by the Maximum velocity ( $U = \omega R$ ).



**Fig. 1: Velocity Profile for  $\eta=0.5$ , (a) Inner cylinder is rotating and Outer is fixed. (b) Outer cylinder is rotating and inner is fixed. (c) Both cylinders are rotating in same direction with same angular velocity. (d) Both cylinders are rotating in opposite direction with same velocity.**

Figure (1), represents the Variation of the Velocity with respect to radial distance for rotation of the either inner or outer cylinder or both cylinder rotates in same and opposite direction for  $\eta=0.5$ .

#### IV. LINEAR STABILITY ANALYSIS

Any Disturbances that can be produced in to the flow field is decomposed in to the large number of the disturbances. If this disturbances are decay with respect to time we can say that flow will steady after the some interval of time and if it is increase with respect to time we can say flow will unsteady.

Normal mode analysis is carried out to obtain a 3D perturbation form of the governing equation which is the sum of the Basic state and 3D Disturbances flow component.

I.e. Instantaneous flow component = Basic state flow + 3D Disturbances component

$$(u_r, u_\theta, u_z, p) = [0, \bar{u}_\theta(r), 0, 0] + Real(u'_r, u'_\theta, u'_z, p') \quad (9)$$

Where,

$$(u'_r, u'_\theta, u'_z, p') = (u_r(r), u_\theta(r), u_z(r), p(r)) e^{i(\alpha\theta + \beta z - \omega t)} \quad (10)$$

Also, superscript in Eq. (9) & (10) represents the disturbances quantity.

So, 3D Perturbation governing equation,

$$-\left\{ \frac{1}{Re} \left[ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{\alpha^2}{r^2} - \beta^2 - \frac{1}{r^2} \right] - \frac{i\alpha}{r} \bar{u}_\theta \right\} \{u'_r\} - \left\{ \frac{2\bar{u}_\theta}{r} - \frac{2i\alpha}{Re * r^2} \right\} \{u'_\theta\} + 0\{u'_z\} + \left\{ \frac{\partial}{\partial r} \right\} \{p'\} = i\omega \{u'_r\} \quad (11)$$

$$\left\{ \frac{\partial \bar{u}_\theta}{\partial r} + \frac{\bar{u}_\theta}{r} - \frac{2i\alpha}{Re * r^2} \right\} \{u'_r\} - \left\{ \frac{1}{Re} \left[ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{\alpha^2}{r^2} - \beta^2 - \frac{1}{r^2} \right] - \frac{i\alpha}{r} \bar{u}_\theta \right\} \{u'_\theta\} + 0\{u'_z\} + \left\{ \frac{i\alpha}{r} \right\} \{p'\} = i\omega \{u'_\theta\} \quad (12)$$

$$\{0\}\{u'_r\} - \{0\}\{u'_\theta\} - \left\{ \frac{1}{Re} \left[ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{\alpha^2}{r^2} - \beta^2 - \frac{1}{r^2} \right] \right\} \{u'_z\} + \{i\beta\}\{p'\} = i\omega\{u'_z\} \quad (13)$$

$$\left\{ \frac{\partial}{\partial r} + \frac{1}{r} \right\} \{u'_r\} + \left\{ \frac{i\alpha}{r} \right\} \{u'_\theta\} + \{i\beta\}\{u'_z\} = 0 \quad (14)$$

Where,  $\omega$  is the circular angular velocity & it is complex quantity ( $\omega = \omega_r + i\omega_i$ ). If  $\omega_i > 0$ , then flow is Unstable, if  $\omega_i < 0$ , then flow is Stable and  $\omega_i = 0$ , then flow is Neutrally Stable. Also,  $\alpha$  &  $\beta$  is the stream wise and Longitudinal wave number. The Reynolds number is to be Critical Reynolds number if  $\omega_i = 0$  in the  $\alpha$  against

### A. Solution Methods

The Chebyshev Spectral Collocation method [11] is used to discretize the governing equation (11)-(14) and cluster the more grid point at the wall of cylinder. Chebyshev polynomial is help to generate the Chebyshev Differentiation Matrix (CDM) by Chebyshev collocation points. The Chebyshev Points or Chebyshev Gauss-Lobatto Points is defined as  $r_i = \cos(i\pi/N)$  where,  $i = (1, N)$

The Validation of the Chebyshev Spectral Collocation method by derivation of the  $\sin x$  using analytically as shown in fig. (2). In Fig. (2), the spacing between two grids points are less at the end as compared to the center. The main objectives of this methods is to discretize the radial distance and produce a finer mesh at the walls of the cylinders in order to an account effect of the disturbances on the stability of the Taylor-Couette flow.

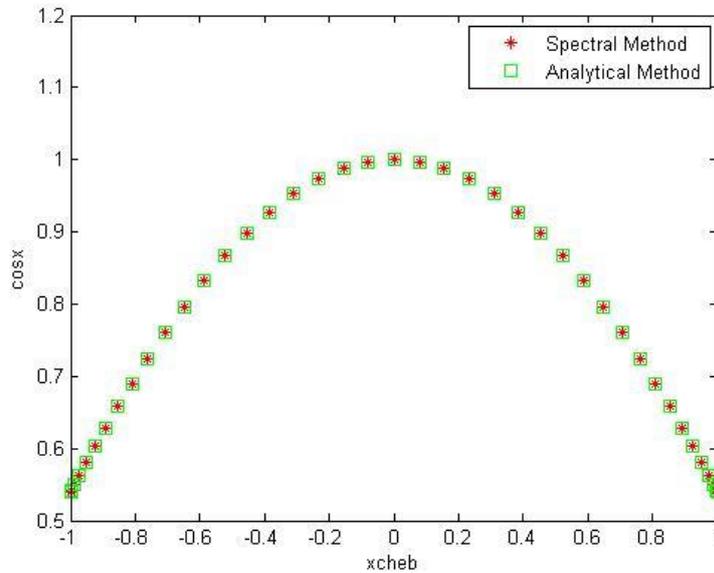


Fig. 2: Validation of the Chebyshev Spectral Collocation Methods by derivation of the  $\sin x$  using analytically and Spectral methods. (41 Chebyshev Points)

### B. Perturbation Boundary Conditions:

The Perturbation Boundary conditions at the walls of the cylinders are,

$$u'_r = 0, \quad u'_\theta = 0, \quad u'_z = 0 \text{ at } r = R_1 \text{ \& \& } R_2 \text{ (No slip Boundary Condition)} \quad (15)$$

The Compatibility Boundary Conditions are,

$$-\left\{ \frac{1}{Re} \left[ \frac{\partial^2 u'_r}{\partial r^2} + \frac{1}{r} \frac{\partial u'_r}{\partial r} \right] \right\} + \left\{ \frac{i\beta}{r} \right\} \{p'\} = 0, \quad -\left\{ \frac{1}{Re} \left[ \frac{\partial^2 u'_r}{\partial r^2} + \frac{1}{r} \frac{\partial u'_r}{\partial r} \right] \right\} + \left\{ \frac{i\alpha}{r} \right\} \{p'\} = 0, \quad (16)$$

$$-\left\{ \frac{1}{Re} \left[ \frac{\partial^2 u'_r}{\partial r^2} + \frac{1}{r} \frac{\partial u'_r}{\partial r} \right] \right\} + \left\{ \frac{\partial}{\partial r} \right\} \{p'\} = 0 \quad (17)$$

The Eigen value and Eigen Spectrum are obtained by the solving the equation (11)-(14) with the help of the Chebyshev Spectral Collocation Methods and appropriate Perturbation Boundary conditions and arranged in the Matrix form,

$$\therefore A\Phi = \lambda B\Phi \tag{18}$$

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{44} & A_{44} \end{bmatrix} \begin{bmatrix} u'_r \\ u'_\theta \\ u'_z \\ p' \end{bmatrix} = i\omega \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \\ B_{31} & B_{32} & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{44} & B_{44} \end{bmatrix} \begin{bmatrix} u'_r \\ u'_\theta \\ u'_z \\ p' \end{bmatrix}$$

### V. RESULTS AND DISCUSSIONS

The Eigen Spectrum for Viscous flow, incompressible between two counter rotating cylinders for various value of the  $\eta$  (0.5&0.99) as shown in Fig. 3. The Eigen Spectrum represents the graph of the imaginary Vs. real part of the Eigen values or angular frequency. The Reynolds number which is assigned as a Critical Re is based on the  $\omega_i=0$ . In this case  $\omega_i$  is considered as an approximate zero if it is  $O(10^{-4})$ . Quick response algorithm is used to generates the Eigen Spectrum for  $\eta=0.5$  & 0.99.

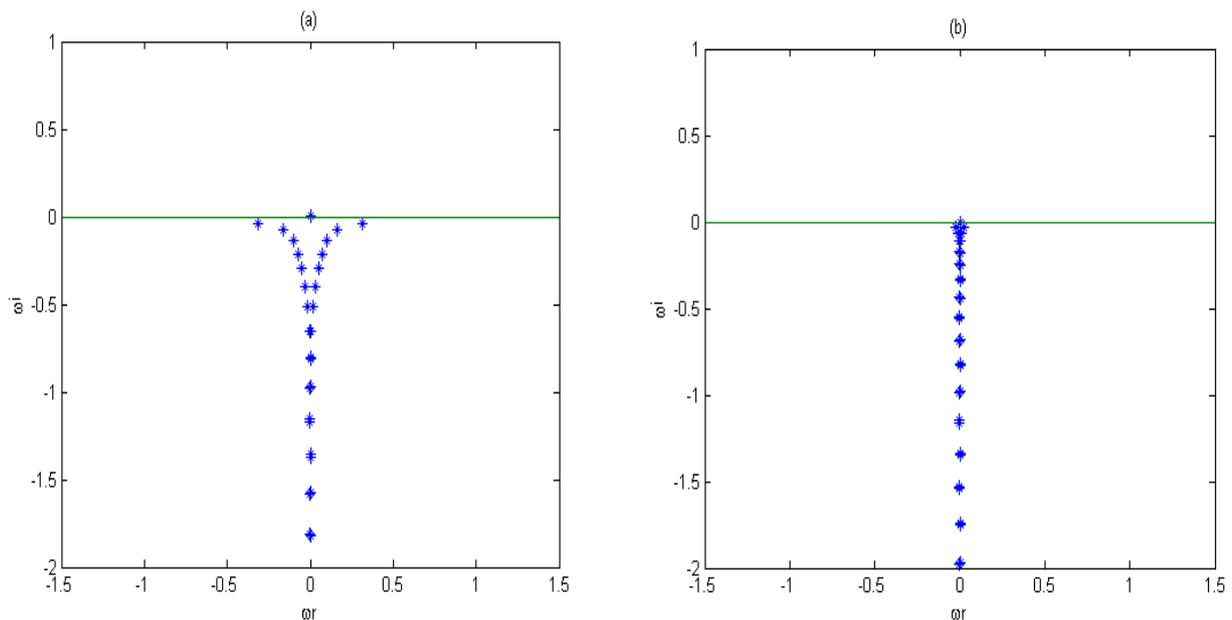


Fig. 3: Eigen Spectrum for Taylor Couette flow for (a)  $\eta=0.5$  & (b)  $\eta=0.99$  with  $\alpha=0$  &  $\beta=\pi/2$ .

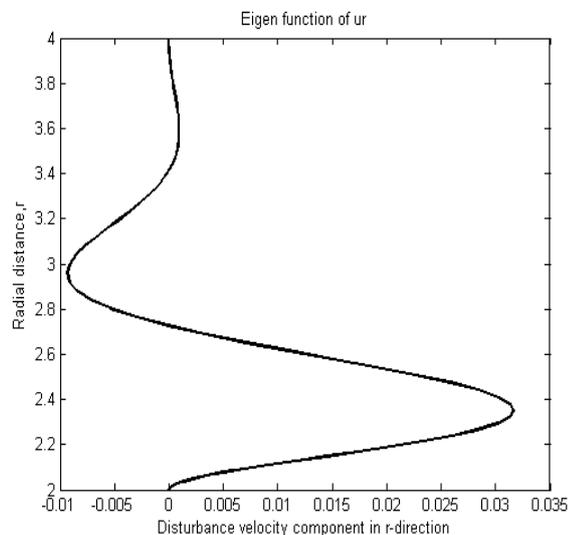


Fig. 4: Radial Disturbances Velocity Component

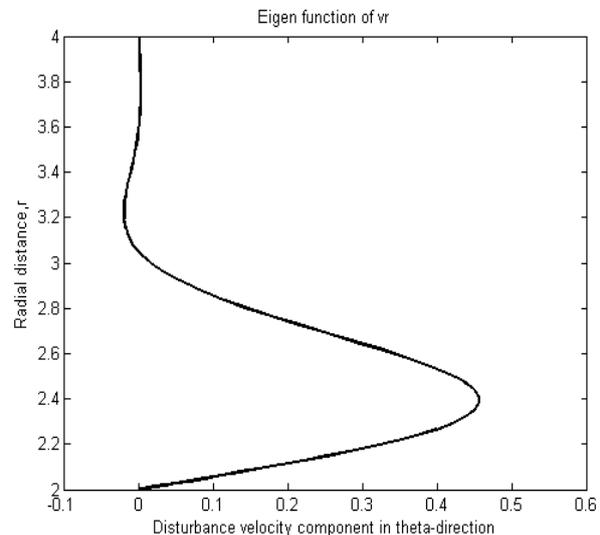


Fig. 5: Tangential Disturbances Velocity Component

Fig. 4-7, represents the Eigen function or Disturbances Velocity component & pressure for  $\eta=0.5$ ,  $\Omega=-1$ ,  $\alpha=0$  &  $\beta=\pi/2$ . Disturbances Velocity components are zero at the inner and outer walls of the cylinders as shown in fig. (4)-(8) that means No-Slip boundary condition is successfully applied at the walls. While the Disturbances pressure is not zero at walls as shown in fig. 7. Critical Reynolds number for  $\eta=0.5$  &  $0.99$  are  $306$  &  $362$  respectively for fixed value of the  $\Omega=-1$ ,  $\alpha=0$  &  $\beta=\pi/2$ . Also, Critical  $re$  is increased with increased in the  $\eta$ .

Now, considering the effect of the Angular Velocity ratio on the Stability of the Taylor-Couette flow and minus sign in Angular Velocity ratio indicates the both cylinders are rotating in opposite direction. Here, two case are consider, first case is to  $\omega_1 > \omega_2$  and second case is  $\omega_1 < \omega_2$ . In first case Angular speed of the inner cylinder is greater than the outer cylinders and with increasing the inner cylinder speed Critical Reynolds number is decreased as shown in the Table-1 for  $\eta=0.5$ ,  $\Omega=-1$ ,  $\alpha=0$  &  $\beta=\pi/2$ . While in second case Angular speed of the outer cylinder is greater than the inner cylinder or reduction in the speed of the inner cylinder leads to increase the Critical  $Re$  as shown in Table-2. That means is stabilize the flow.

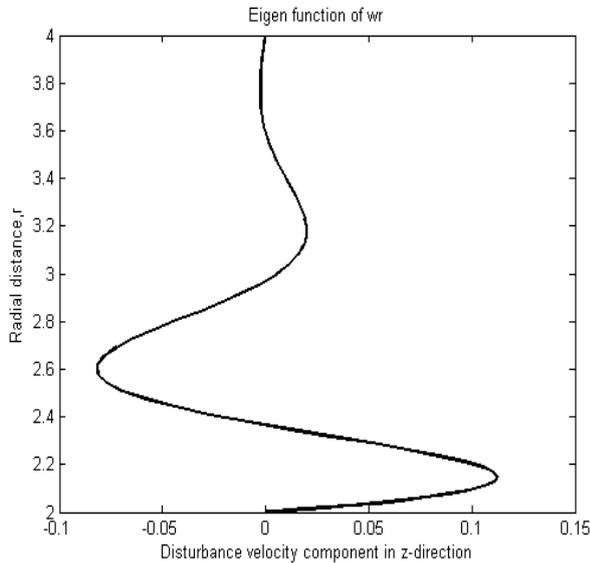


Fig. 6: Axial Disturbances Velocity Component

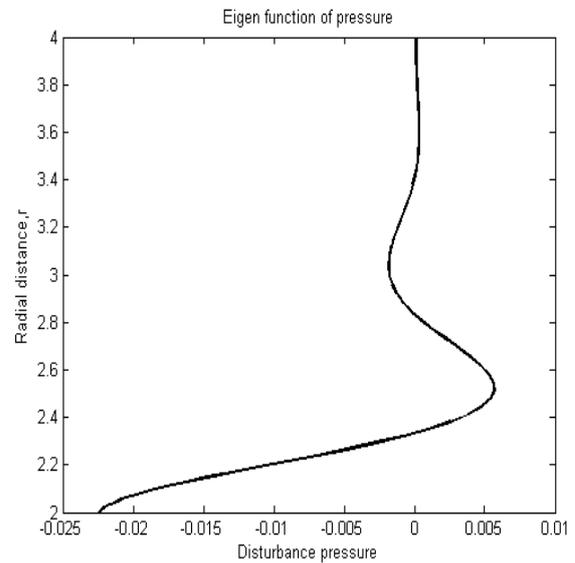


Fig. 7: Pressure Disturbances

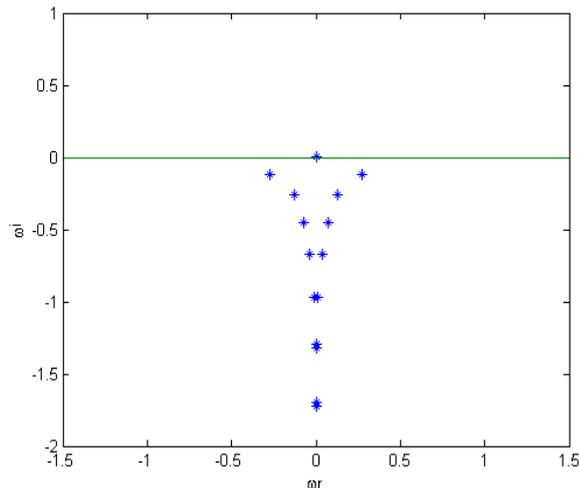
$\Omega(\omega_1 > \omega_2)$	Critical $Re$	Largest $\omega_i$
-0.6	93	3.60e-004
-0.7	131	3.9e-004
-0.8	179	8.9e-004
-0.9	234	5.0e-004
-1	306	1.37e-004

Table 1: Calculation of the Critical  $Re$  and corresponding Largest  $\omega_i$  for various values of the  $\Omega(\omega_1 > \omega_2)$  and fixed value of the  $\eta=0.5$ ,  $\Omega=-1$ ,  $\alpha=0$  &  $\beta=\pi/2$ .

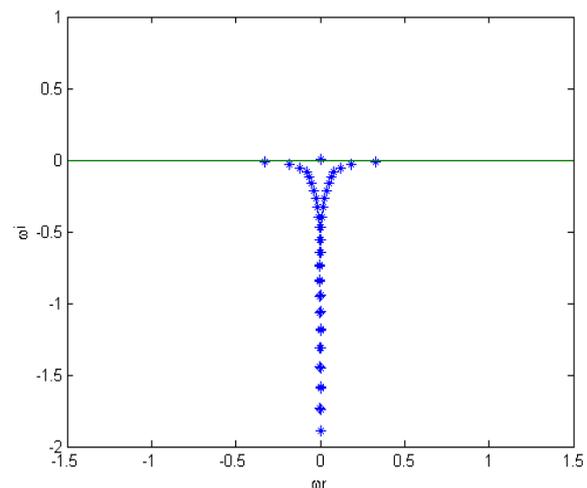
$\Omega(\omega_1 < \omega_2)$	Critical $Re$	Largest $\omega_i$
-1	306	1.37e-004
-1.1	391	6.6e-005
-1.2	498	9.1e-004
-1.3	607	1.8e-004
-1.4	752	5.4e-004

Table 2: Calculation of the Critical  $Re$  and corresponding Largest  $\omega_i$  for various values of the  $\Omega(\omega_1 < \omega_2)$  and fixed value of the  $\eta=0.5$ ,  $\Omega=-1$ ,  $\alpha=0$  &  $\beta=\pi/2$ .

Fig. (8) & (9) represent the Eigen Spectrum for Taylor-Couette flow for  $\Omega=-0.6$  &  $1.4$  for fixed value of the  $\eta=0.5$ ,  $\alpha=0$  &  $\beta=\pi/2$  respectively.



**Fig. 8: Eigen Spectrum for  $\eta=0.5, \Omega=-0.6, \alpha=0$  &  $\beta=\pi/2$ .**



**Fig. 9: Eigen Spectrum for  $\eta=0.5, \Omega=-1.4, \alpha=0$  &  $\beta=\pi/2$ .**

### CONCLUSION

In this paper we study the temporal stability of the viscous, incompressible flow between two opposite rotating cylinders. The main aim is to find the Critical Re for different values of the radius ratio and speed ratio. Critical Re are 306 & 362 for  $\eta=0.5$  &  $0.99$  respectively. In that case both cylinders are Counter rotating,  $\Omega=-1$ . So we compute the Critical Re with increase and decrease the value of the speed ratio,  $\Omega$ . Angular velocity ratio is the Ratio of the inner cylinder to outer cylinder speed, if we increase the inner cylinder speed ( $\Omega<-1$ ) than Critical Re is decreased and if we decrease the inner cylinder speed or increase the outer cylinder speed ( $\Omega>-1$ ) than Critical Re is increased. That means  $\Omega<-1$  leads to destabilize the flow while  $\Omega>-1$  leads to stabilize the flow.

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