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Fixed Point Theorem Involving Occasionally Weakly Compatible Maps

Dr.M.RamanaReddy,

Assistant Professor of Mathematics
Sreenidhi Institute of Science and technology,Hyderabad

Abstact: *In this article we proved a generalized common fixed point theorem involving occasionally weakly compatible maps*

Keywords: *Weakly compatible, point of coincidence, complete metric space*

I.Introduction

The study of fixed point theorems and common fixed point theorems satisfying contractive type conditions has been a very active field of research activity during the last decades. In 1972 a new geometrically concept which is different from Banach [1] and Kannan [2] for contraction type mapping was introduced by Chatterjee [3] which gives a new direction to the study of the fixed point theory. Chatarjee [3] gives following contraction principle:

There exists a number α where $0 < \alpha < 1$ such that for each $x, y \in X$

$$d(Tx, Ty) \leq \alpha [\, d(x, Ty) + d(y, Tx)]$$

In 1978, Fisher B. [4] generalized the result of Kannan by choosing α which as follows:

$$d(Tx, Ty) \leq \alpha [d(x, Ty) + d(Tx, y)]$$

For all $x, y \in X$ and $0 \leq \alpha \leq \frac{1}{2}$ then T has unique fixed point in X.

A number of these results dealt with fixed points for more than one map. In some cases commutatively between the maps was required in order to obtain a common fixed point. First of all Jungck [5] introdicued the notion of commutative mapping which is also known as commuting mapping and fixed common fixed point result for two different self mappings. Sessa [6] coined the term weakly commuting. Jungck [5] generalized the notion of weak commutativity by introducing the concept of compatible maps and then weakly compatible maps [7]. There are examples that show that each of these generalizations of commutativity is a proper extension of the previous definition. Also, during this time a number of researchers established fixed point theorems for pair of maps.

The aim of this article is to obtain some fixed point theorem involving occasionally weakly compatible maps in the setting of symmetric space satisfying a rational contractive condition. Our results complement, extend and unify several well known comparable results.

II.Preliminaries

Definition 2.1 Let S and T are self maps of a metric space X. If $w = Sx = Tx$ for some $x \in X$, then x is called a coincidence point of S and T, and w is called a point of coincidence of S and T.

Definition 2.2 Let S and T are self maps of a metric space X, then S and T are said to be weakly compatible if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0$$

whenever $\{x_n\}$ is sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x$$

for some $x \in X$.

Definition 2.3 Let S and T are self maps of a metric space X, then S and T are said to be weakly compatible if they commute at their coincidence points; i.e. if $Tx = Sx$ for some $x \in X$ then $TSx = STx$.

Definition 2.4 Let Φ be the set of real functions

$$\phi(t_1, t_2, t_3, t_4, t_5): [0, \infty)^5 \to [0, \infty)$$

satisfying the following conditions:

i. ϕ is non increasing in variables t_4 and t_5.
ii. There is an $h_1 > 0$ *and* $h_2 > 0$ such that $h = h_1 h_2 < 1$ and if $u \geq 0$ and $v \geq 0$ satisfying

a. $u \le \phi(v, v, u, u+v, 0)$ or $u \le \phi(v, u, v, u+v, 0)$
Then we have $u \le h_1 v$.
And if $u \ge 0, v \ge 0$ satisfy

b. $u \le \phi(v, v, u, 0, u+v)$ or $u \le \phi(v, u, v, 0, u+v)$
Then we have $u \le h_2 v$.
If $u \ge 0$ is such that

$$u \le \phi(u, 0,0, u, u) \text{ or } u \le \phi(0, u, 0,0, u) \text{ or } u \le \phi(0,0, u, u, 0)$$

Then $u = 0$.

III. Main Result

Theorem 3.1: Let A, B, S, T be continuous self mappings defined on the complete metric space X into itself satisfies the following conditions:
(i) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$
(ii) if one of $A(X)$, $B(X)$, $S(X)$, $T(X)$ is complete subspace of X.
(iii) The pair $\{A, S\}$ and $\{B, T\}$ are weakly compatible.

$$\text{(iv)}\quad d(Ax, By) \le \alpha \max \left\{ \begin{array}{c} \frac{(d(Ax,Sx))^2 + (d(By,Ty))^2}{d(Ax,Sx)+d(By,Ty)}, \\ \frac{(d(Ax,Ty))^2 + (d(By,Sx))^2}{d(Ax,Ty)+d(By,Sx)}, \\ \frac{(d(Ax,Sx))^2 + (d(Ax,Ty))^2}{d(Ax,Sx)+d(Ax,Ty)}, \\ \frac{(d(By,Sx))^2 + (d(By,Ty))^2}{d(By,Sx)+d(By,Ty)} \end{array} \right\}$$

For all $x, y \in X$, $(x \ne y)$ and for non negative $\alpha \in [0,1)$. Then A, B, S, T have unique common fixed point in X.

Proof: For any arbitrary x_0 in X we define the sequence $\{x_n\}$ and $\{y_n\}$ in X such that

$$Ax_{2n} = Tx_{2n+1} = y_{2n} \text{ and } Bx_{2n+1} = Sx_{2n+2} = y_{2n+1}$$

for all n= 0, 1, 2, …..
On taking $y_{2n} \ne y_{2n+1}$

$$d(y_{2n}, y_{2n+1}) = d(Ax_{2n}, Bx_{2n+1})$$

From (iv) we have

$$d(Ax_{2n}, Bx_{2n+1}) \le \alpha \max \left\{ \begin{array}{c} \frac{(d(Ax_{2n},Sx_{2n}))^2 + (d(Bx_{2n+1},Tx_{2n+1}))^2}{d(Ax_{2n},Sx_{2n})+d(Bx_{2n+1},Tx_{2n+1})}, \\ \frac{(d(Ax_{2n},Tx_{2n+1}))^2 + (d(Bx_{2n+1},Sx_{2n}))^2}{d(Ax_{2n},Tx_{2n+1})+d(Bx_{2n+1},Sx_{2n})}, \\ \frac{(d(Ax_{2n},Sx_{2n}))^2 + (d(Ax_{2n},Tx_{2n+1}))^2}{d(Ax_{2n},Sx_{2n})+d(Ax_{2n},Tx_{2n+1})}, \\ \frac{(d(Bx_{2n+1},Sx_{2n}))^2 + (d(Bx_{2n+1},Tx_{2n+1}))^2}{d(Bx_{2n+1},Sx_{2n})+d(Bx_{2n+1},Tx_{2n+1})} \end{array} \right\}$$

$$d(y_{2n}, y_{2n+1}) \le \alpha \max \left\{ \begin{array}{c} \frac{(d(y_{2n},y_{2n-1}))^2 + (d(y_{2n+1},y_{2n}))^2}{d(y_{2n},y_{2n-1})+d(y_{2n+1},y_{2n})}, \\ \frac{(d(y_{2n},y_{2n}))^2 + (d(y_{2n+1},y_{2n-1}))^2}{d(y_{2n},y_{2n})+d(y_{2n+1},y_{2n-1})}, \\ \frac{(d(y_{2n},y_{2n-1}))^2 + (d(y_{2n},y_{2n}))^2}{d(y_{2n},y_{2n-1})+d(y_{2n},y_{2n})}, \\ \frac{(d(y_{2n+1},y_{2n-1}))^2 + (d(y_{2n+1},y_{2n}))^2}{d(y_{2n+1},y_{2n-1})+d(y_{2n+1},y_{2n})} \end{array} \right\}$$

$$d(y_{2n}, y_{2n+1}) \le \alpha \max \left\{ \begin{array}{c} (d(y_{2n}, y_{2n-1}) + d(y_{2n+1}, y_{2n})), \\ (d(y_{2n}, y_{2n-1}) + d(y_{2n+1}, y_{2n})), \\ d(y_{2n}, y_{2n-1}), \\ (d(y_{2n}, y_{2n-1}) + 2d(y_{2n+1}, y_{2n})) \end{array} \right\}$$

$$(1-\alpha)d(y_{2n}, y_{2n+1}) \leq \alpha\, d(y_{2n}, y_{2n-1})$$
$$d(y_{2n}, y_{2n+1}) \leq \frac{\alpha}{1-\alpha}\, d(y_{2n}, y_{2n-1})$$

Let us denote $\frac{\alpha}{1-\alpha} = k$,

$$d(y_{2n}, y_{2n+1}) \leq k\, d(y_{2n}, y_{2n-1})$$

Similarly we can show that

$$d(y_{2n}, y_{2n-1}) \leq k^2\, d(y_{2n-2}, y_{2n-1})$$

Processing the same way we can write,

$$d(y_{2n}, y_{2n-1}) \leq k^n\, d(y_0, y_1)$$

for any integer m we have

$$d(y_{2n}, y_{2n+m}) \leq d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}) + \dots\dots\dots + d(y_{2n+m-1}, y_{2n+m})$$
$$d(y_{2n}, y_{2n+m}) \leq k^n . d(y_0, y_1) + k^{n+1} . d(y_0, y_1) + \dots\dots\dots + k^{n+m} . d(y_0, y_1)$$
$$d(y_{2n}, y_{2n+m}) \leq k^n [1 + k + k^2 + \cdots \dots\dots + k^m] . d(y_0, y_1)$$
$$d(y_{2n}, y_{2n+m}) \leq \frac{k^n}{1-k} . d(y_0, y_1)$$

as $n \to \infty$ gives that

$$d(y_{2n}, y_{2n+m}) \to 0$$

Thus $\{y_{2n}\}$ is a Cauchy sequence in X. Since T(X) is complete subspace of X then the subsequence $y_{2n} = Tx_{2n+1}$ is Cauchy sequence in T(X) which converges to the some point say u in X. Let $v \in T^{-1}u$ then $Tv = u$. Since $\{y_{2n}\}$ is converges to u and hence $\{y_{2n+1}\}$ also converges to same point u .

we set $x = x_{2n}$ and $y = v$ in (iv)

$$d(Ax_{2n}, Bv) \leq \alpha \max \left\{ \begin{array}{c} \frac{(d(Ax_{2n},Sx_{2n}))^2 + (d(Bv,Tv))^2}{d(Ax_{2n},Sx_{2n}) + d(Bv,Tv)}, \\ \frac{(d(Ax_{2n},Tv))^2 + (d(Bv,Sx_{2n}))^2}{d(Ax_{2n},Tv) + d(Bv,Sx_{2n})}, \\ \frac{(d(Ax_{2n},Sx_{2n}))^2 + (d(Ax_{2n},Tv))^2}{d(Ax_{2n},Sx_{2n}) + d(Ax_{2n},Tv)}, \\ \frac{(d(Bv,Sx_{2n}))^2 + (d(Bx_{2n+1},Tv))^2}{d(Bv,Sx_{2n}) + d(Bx_{2n+1},Tv)} \end{array} \right\}$$

as $n \to \infty$

$$d(u, Bv) \leq \alpha\, d(u, Bv)$$

which contradiction

implies that $Bv = u$ also $B(X) \subset S(X)$ so $Bv = u$ implies that $u \in S(X)$.

Let $w \in S^{-1}(X)$ then $w = u$ setting $x = w$ and $y = x_{2n+1}$ in (iv) we get

$$d(Ax_{2n}, Bv) \leq \alpha \max \left\{ \begin{array}{c} \frac{(d(Aw,Sw))^2 + (d(Bx_{2n+1},Tx_{2n+1}))^2}{d(Aw,Sw) + d(Bx_{2n+1},Tx_{2n+1})}, \\ \frac{(d(Aw,Tx_{2n+1}))^2 + (d(Bx_{2n+1},Sw))^2}{d(Aw,Tx_{2n+1}) + d(Bx_{2n+1},Sw)}, \\ \frac{(d(Aw,Sw))^2 + (d(Aw,Tx_{2n+1}))^2}{d(Aw,Sw) + d(Aw,Tx_{2n+1})}, \\ \frac{(d(Bx_{2n+1},Sw))^2 + (d(Bx_{2n+1},Tx_{2n+1}))^2}{d(Bx_{2n+1},Sw) + d(Bx_{2n+1},Tx_{2n+1})} \end{array} \right\}$$

as $n \to \infty$

$$d(Aw, u) \leq \alpha\, d(Aw, u)$$

which contradiction

implies that, $Aw = u$ this means $Aw = Sw = Bv = Tv = u$.

since $Bv = Tv = u$ so by weak compatibility of (B, T) it follows that, $BTv = TBv$ and so we get

$$Bu = BTv = TBv = Tu .$$

Since $Aw = Sw = u$ so by weak compatibility of (A, S) it follows that $SAw = ASw$ and So we get

$$Au = ASw = SAw = Su$$

Thus from (iv) we have

$$d(Aw, Bu) \leq \alpha \max \left\{ \begin{array}{l} \frac{(d(Aw,Sw))^2 + (d(Bu,Tu))^2}{d(Aw,Sw)+d(Bu,Tu)}, \\ \frac{(d(Aw,Tu))^2 + (d(Bu,Sw))^2}{d(Aw,Tu)+d(Bu,Sw)}, \\ \frac{(d(Aw,Sw))^2 + (d(Aw,Tu))^2}{d(Aw,Sw)+d(Aw,Tu)}, \\ \frac{(d(Bu,Sw))^2 + (d(Bu,Tu))^2}{d(Bu,Sw)+d(Bu,Tu)} \end{array} \right\}$$

$$d(u, Bu) \leq \alpha d(u, Bu)$$

which contradiction
implies that $Bu = u$.
Similarly we can show $Au = u$ by using (iv). Therefore

$$u = Au = Bu = Su = Tu.$$

Hence the point u is common fixed point of A, B, S, T.
If we assume that $S(X)$ is complete then the argument analogue to the previous completeness argument proves the theorem. If $A(X)$ is complete then $u \in A(X) \subset T(X)$. similarly if $B(X)$ is complete then $u \in B(X) \subset S(X)$. This complete prove of the theorem.

Uniqueness Let us assume that z is another fixed point of A, B, S, T in X different from u. i.e. $u \neq z$ then

$$d(u, z) = d(Au, Bz) \text{ from (iv) we get}$$
$$d(u, z) \leq \alpha\, d(u, z)$$

which contradiction the hypothesis . Hence u is unique common fixed point of A, B, S, T in X.

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