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OSCILLATION AND STABILITY IN A MASS SPRING SYSTEM

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Abstract — *This paper examines the dynamical behavior of Damped and Undamped motions of mass spring system represented by Homogeneous Differential Equations as well as Discrete Fractional order Equations. For each system the equilibrium position is established and the nature of the system is analyzed. Also time line plots and phase diagrams for appropriate numerical values are produced.*

Keywords- *Mass Spring, Homogeneous, Oscillation, Stability, Fractional differential equations.*

[2]1 ESTABLISHMENT OF MASS SPRING SYSTEM

Mass spring system is one of the most important applications of second order differential equations [2, 5, 8]. Mass spring system is described as

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt} + F(t)$$

This derived from Newton's second law of motion ([2, 5]). Here m is mass; $\frac{d^2x}{dt^2}$ is acceleration; $-kx$ is restoring force (k is spring constant); $-\beta\frac{dx}{dt}$ is damping force (β is damping constant); and $F(t)$ denotes external forces acting on the mass, which depends on both mass and time. Thus

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = F(t) \tag{1}$$

The equation is homogeneous or nonhomogeneous depending on whether forces other than the spring and damping forces act on the mass. In this paper we are considering only two forces acting on the mass, ignoring the external forces.

[2]2 MASS SPRING SYSTEM WITH DAMPING VIBRATIONS

All vibrations are subject to damping of some sort, when no other forces act on the mass, besides the spring, and gravity for vertical oscillations. Differential equation (1) becomes homogeneous of the form,

$$\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + \frac{k}{m}x = 0 \tag{2}$$

Solving (2), we get the roots $r_{1,2} = \frac{-\beta \pm \sqrt{\beta^2 - 4mk}}{2m}$. We shall show that three types of motion can occur [2, 5];

1. Over damped if $\beta^2 - 4mk > 0$ (the roots are real and distinct).
2. Critically damped if $\beta^2 - 4mk = 0$ (the roots are real and equal).
3. Under damped if $\beta^2 - 4mk < 0$ (the roots are imaginary).

2.1. System of Differential Equations

Second order homogeneous differential equation is converted into a system of first order differential equations by taking $\frac{dx}{dt} = y(t)$ [4]. Equation (2) takes the form

$$\begin{aligned} \frac{dx}{dt} &= y(t) \\ \frac{dy}{dt} &= -\frac{\beta}{m}y(t) - \frac{k}{m}x(t) \end{aligned} \tag{3}$$

The system (3) has only one fixed point namely $(0,0)$ and the Jacobian Matrix J for the system (3) is [4, 7]

$$J(x,y)=\begin{bmatrix}0 & 1\\ -\frac{k}{m} & -\frac{\beta}{m}\end{bmatrix} \tag{4}$$

The eigen values of (3) are $\lambda_{1,2}=-\frac{\beta}{2m}\pm\frac{1}{2m}\sqrt{\beta^2-4mk}$. Next we discuss the types of motion with illustration.

2.1.1. Over damped Motion.

The system (3) is over damped if $\beta^2-4mk>0$, so that $\beta^2>4mk$. Let $m=1;\beta=4;$ and $k=2$ with the initial condition $x(0)=0.5; y(0)=0.5$. From the Jacobian matrix (4), we obtain the roots $\lambda_1=-0.5858$ and $\lambda_2=-3.4142$, which are real and distinct. In figure - 1 is *overdamped motion;* damping is so large that oscillations are completely eliminated. The mass simply returns to the equilibrium position without passing through it.

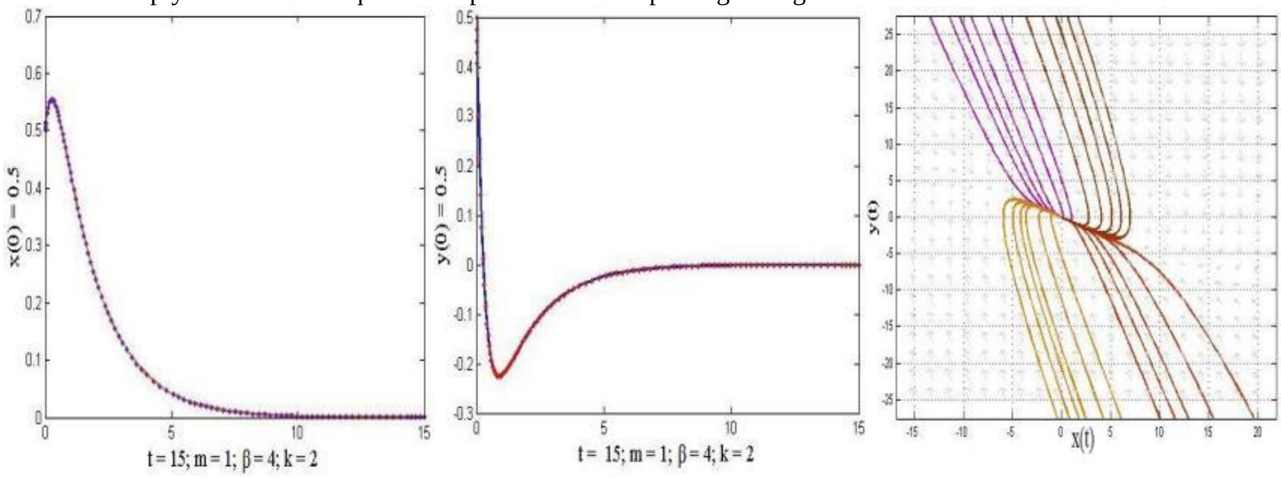


Figure 1. Over damped Motion of (3)

2.1.2. Critically Damped Motion.

The system (3) is critically damped if $\beta^2-4mk=0$, so that $\beta^2=4mk$. Taking $m=1;\beta=4;$ and $k=4$ with the initial condition $x(0)=0.5; y(0)=0.5$. Using the Jacobian matrix (4), the roots are $\lambda_{1,2}=-2$, which are real and equal. In figure - 2 shows *critically damped motion;* Once again no oscillations occur. This situation forms the division between over damped and under damped motion.

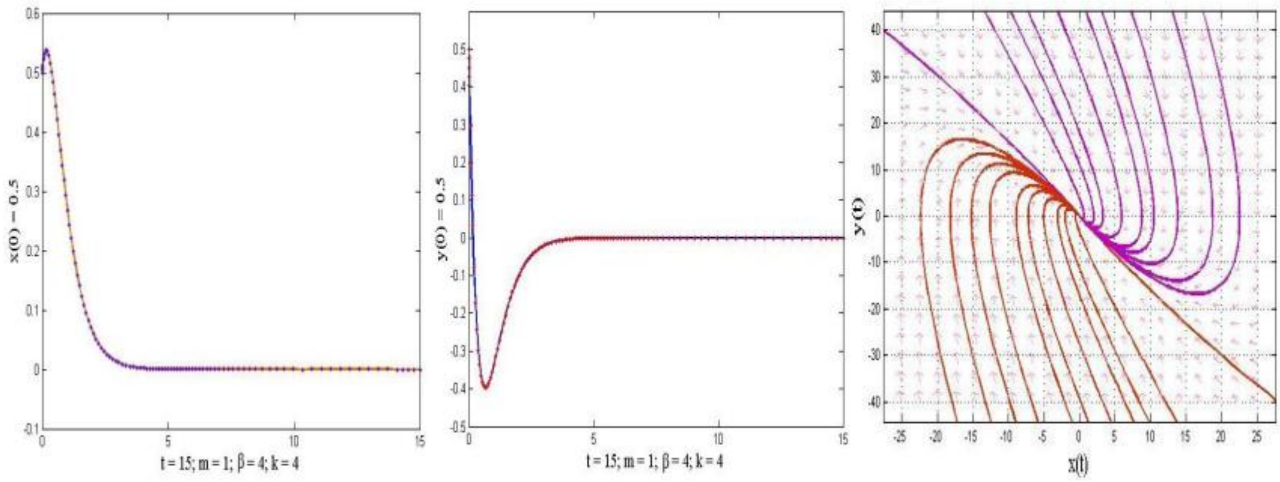


Figure 2. Critically damped for the system (3)

2.1.3. Under damped Motion.

The system (3) is under damped if $\beta^2-4mk<0$, so that $\beta^2<4mk$. Considering $m=1;\beta=0.2;$ and $k=2$ with the initial condition $x(0)=0.5; y(0)=0.5$, the roots are $\lambda_{1,2}=-0.1\pm i1.4101$, which is imaginary. In figure - 3, we get *underdamped motion;* we have damped oscillation and the motion goes to 0 when the time increases.

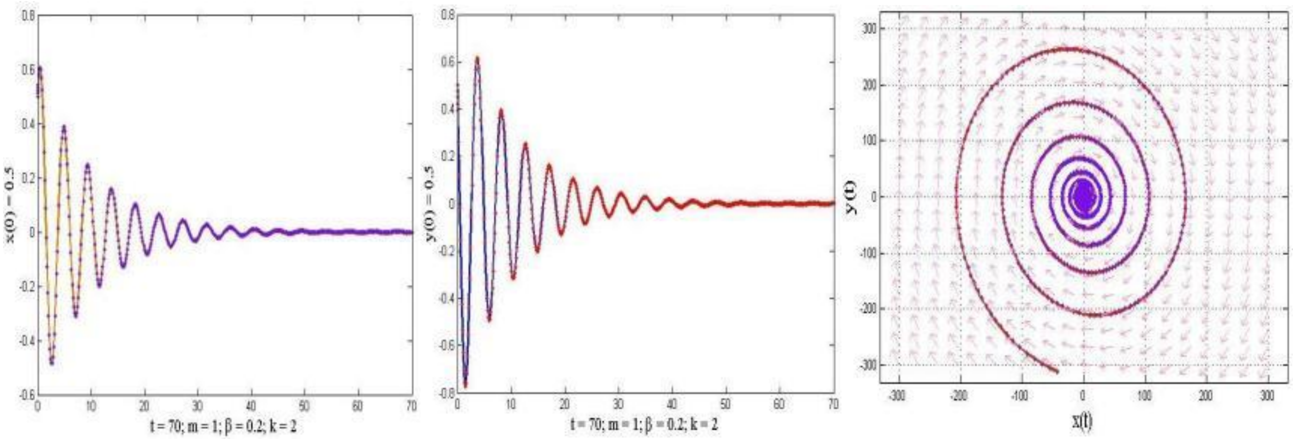


Figure 3. Underdamped for the system (3)

2.2. Discrete Fractional Order System.

Let us consider the system of fractional differential equations as

$$D^{\alpha}x(t) = y(t)$$
$$D^{\alpha}y(t) = -\frac{\beta}{m}y(t) - \frac{k}{m}x(t)$$

where α is the fractional order. [1, 3, 6] When we apply the process of discretization, we get its discrete version in the form,

$$\begin{aligned} x(t+1) &= x(t) + \frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[y(t)\right] \\ y(t+1) &= y(t) - \frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[\frac{\beta}{m}y(t) + \frac{k}{m}x(t)\right] \end{aligned} \tag{5}$$

The fixed point of the system (5) is $(0,0)$ and the Jacobian Matrix J for the system (5) is

$$J(x,y) = \begin{bmatrix} 1 & s \\ -\frac{sk}{m} & 1-\frac{s\beta}{m} \end{bmatrix} \tag{6}$$

where $s = \frac{h^{\alpha}}{\Gamma(1+\alpha)}$, the eigen values of the system (5) are $\lambda_{1,2} = 1 - \frac{\beta s}{2m} \pm \frac{s}{2m}\sqrt{\beta^2 - 4mk}$. Now we investigate the types of motion and plot the numerical simulation with different parameters.

2.2.1. Over damped Motion.

The system (5) is over damped if $\beta^2 - 4mk > 0$, so that $\beta^2 > 4mk$. Let $\alpha = 0.8; h = 0.04; m = 1; \beta = 4;$ and $k = 2$ with the initial condition $x(0) = 0.5; y(0) = 0.5.$ Using the Jacobian matrix (6), the roots are $\lambda_1 = 0.9521$ and $\lambda_2 = 0.7211,$ which is real and distinct. In figure - 4 is *Overdamped motion.*

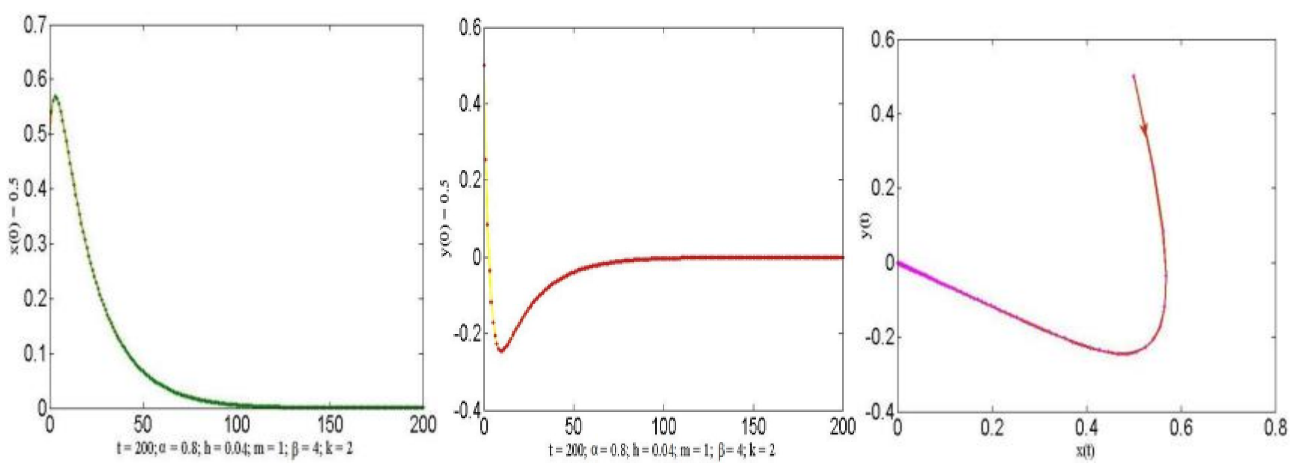


Figure 4. Over damped Motion for (5)

2.2.2. Critically Damped Motion.

The system (5) is critically damped if $\beta^2 - 4mk = 0$, so that $\beta^2 = 4mk$. Taking the values $\alpha = 0.8; h = 0.04;\ m = 1; \beta = 4;$ and $k = 4$ with the initial condition $x(0) = 0.5; y(0) = 0.5$. Using the Jacobian matrix (6), the roots are $\lambda_{1,2} = 0.8366$, which is real and equal. In figure - 5 is *critically damped motion.*

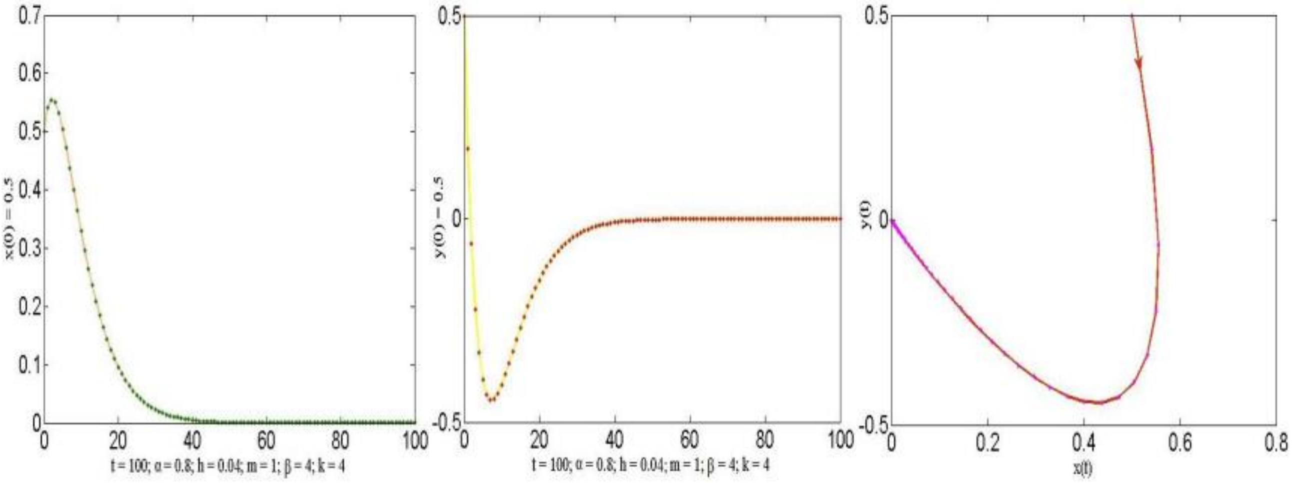


Figure 5. Critically damped Motion for (5)

2.2.3. Under damped Motion.

The system (5) is under damped if $\beta^2 - 4mk < 0$, so that $\beta^2 < 4mk$. Considering $\alpha = 0.8; h = 0.04; m = 1;$ $\beta = 0.2;$ and $k = 2$ with the initial $x(0) = 0.5; y(0) = 0.5$. The roots are $\lambda_{1,2} = 0.9918 \pm i0.1153$, which is imaginary. In figure – 6 is *underdamped motion.*

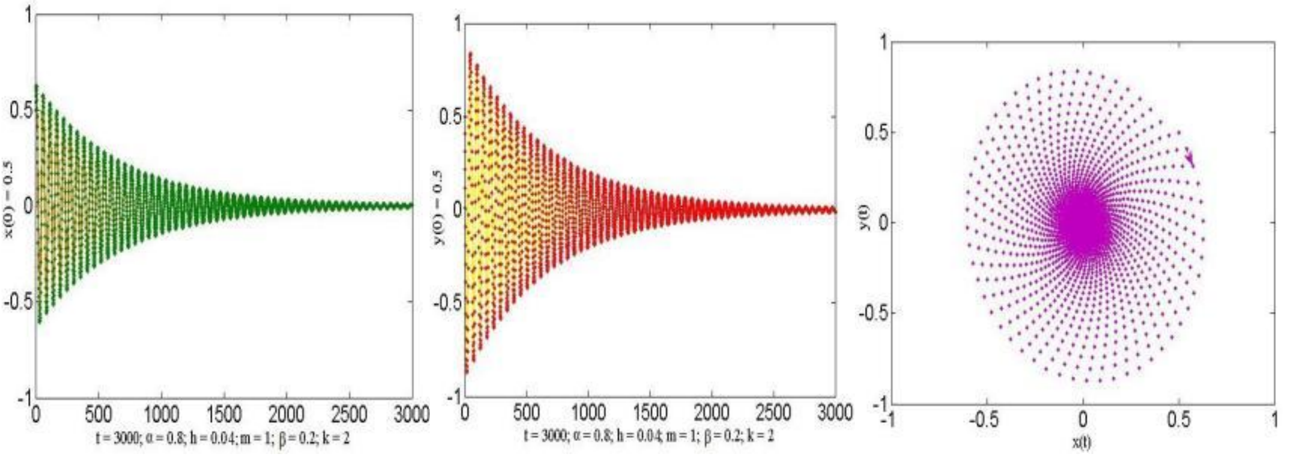
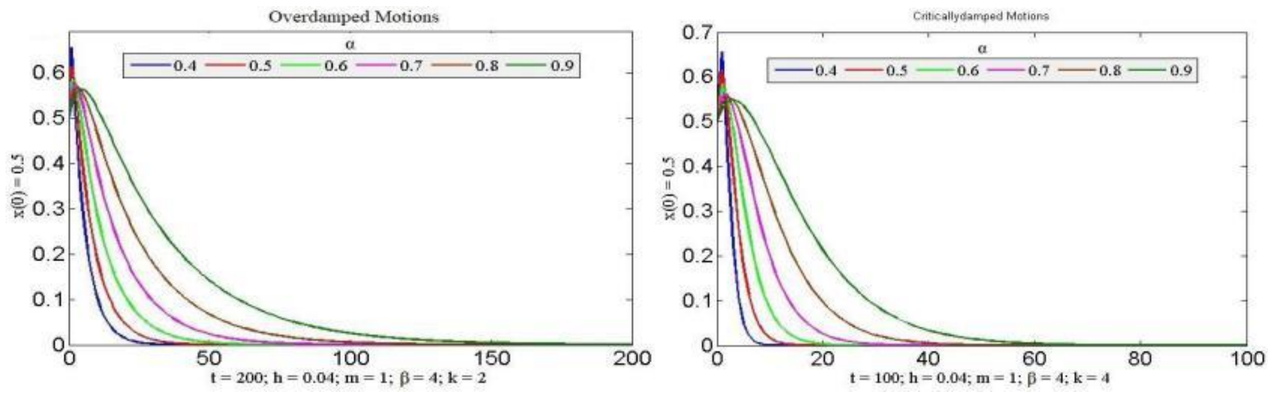


FIGURE 6. Underdamped for the system (5)

In Figures 7, we execute the graph of the dynamical system for different orders of the fractional derivative; namely, $\alpha \in (0.4 - 0.9)$.



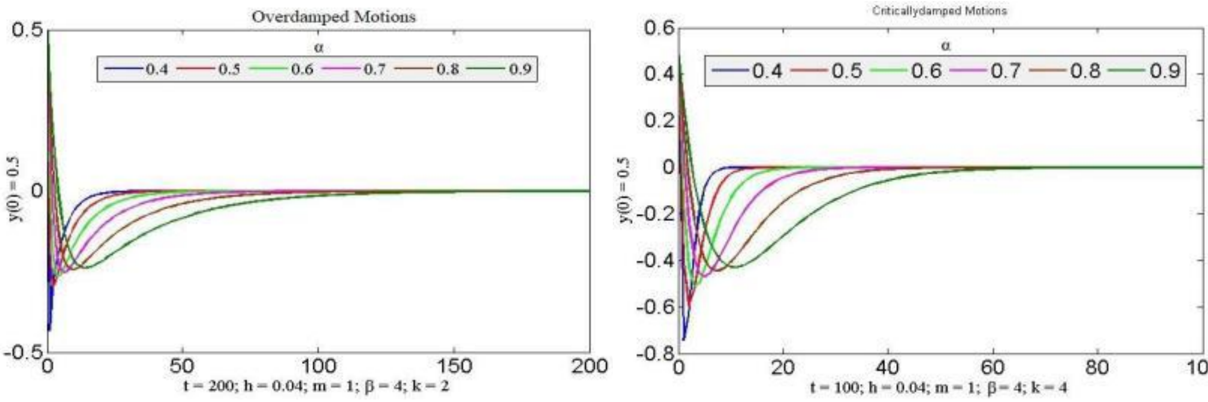


Figure 7. Overdamped and Crically damped motions for the system (5) with various fractional order α 's

In figure – 8, exhibits the underdamped motions for the system (5) with various fractional order $\alpha \in [0.7, 0.8]$. Fractional order α's from 0.7 – 0.72, presents the unbounded oscillations followed by the uniform oscillation when $\alpha = 0.72$, and finally the system attains stability for $\alpha \in [0.78, 0.8]$.

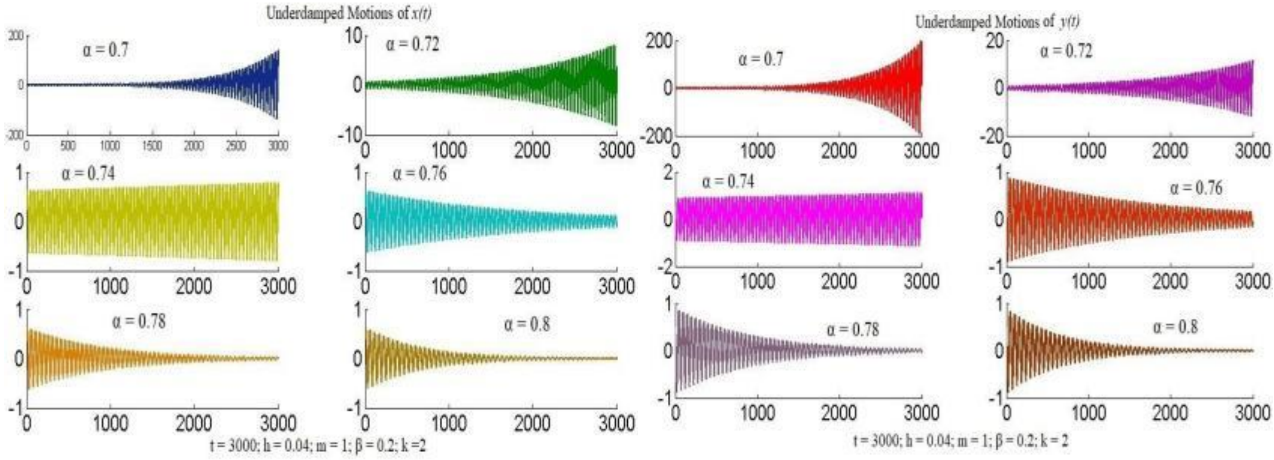


FIGURE 8. Underdamped motions for the system (5) with various fractional order α 's

III. MASS SPRING SYSTEM WITH UNDAMPING VIBRATIONS

In this section, we consider only spring force acting on the mass and ignore the damping forces, which of the form

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \tag{7}$$

Solving the equation (7), we get the roots $r_{1,2} = \pm i\sqrt{\frac{k}{m}}$. Now the equation (7) is change into the system of differential equations given by

$$\begin{aligned} \frac{dx}{dt} &= y(t) \\ \frac{dy}{dt} &= -\frac{k}{m}x(t) \end{aligned} \tag{8}$$

The fixed point of the system (8) is $(0,0)$ and the Jacobian matrix of the system is

$$J(x,y) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} \tag{9}$$

The eigen valus are $\lambda_{1,2} = \pm i\sqrt{\frac{k}{m}}$. Taking the suitable parametric values $m = 1; k = 36$ with the initial conditions $x(0) = 0.5; y(0) = 0.5,$ we get the eigen values $\pm 6i$. The mass oscillates about its equilibrium position forever. This is a direct result of the fact that damping has been ignored. The mass oscillates up and down after a long time period and it is reduced to a stable position when the springs weaken, see figure - 9.

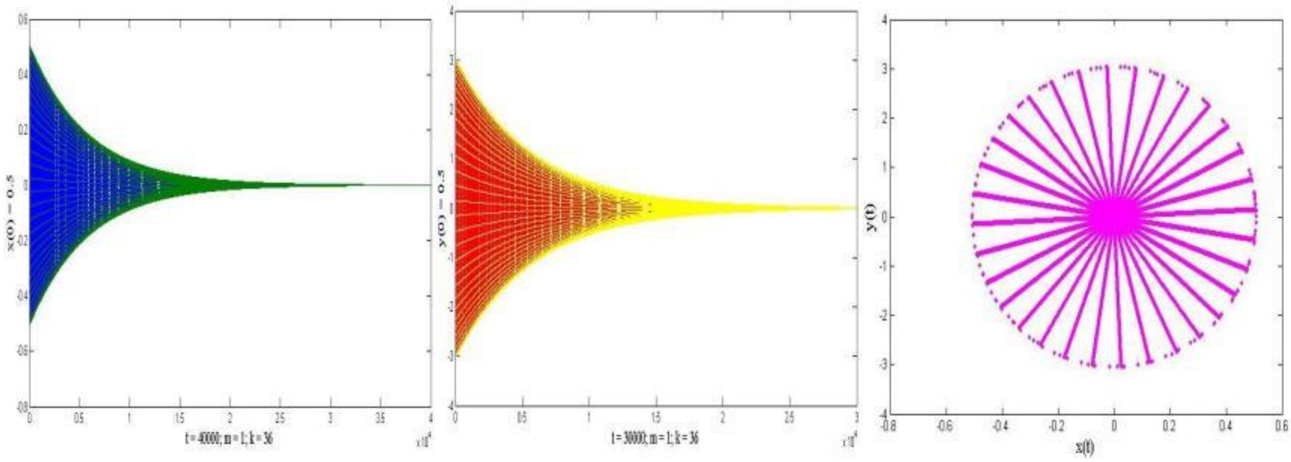


Figure 9. Mass Spring system (8) for Free Damped Motions

Let us consider the system fractional order differential equations of the form

$$D^{\alpha} x(t) = y(t)$$

$$D^{\alpha} y(t) = -\frac{k}{m} x(t)$$

α is the fractional order. Now using the discretization process, the above fractional order differential equations is modified into the form of system of discrete fractional order equations,

$$\begin{aligned} x(t+1) &= x(t) + \frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[y(t)\right] \\ y(t+1) &= y(t) - \frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[\frac{k}{m}x(t)\right] \end{aligned} \qquad (10)$$

The fixed point (10) is $(0,0)$ and the Jacobian Matrix J for (10) is

$$J(x,y) = \begin{bmatrix} 1 & s \\ -\frac{sk}{m} & 1 \end{bmatrix} \qquad (11)$$

where $s = \frac{h^{\alpha}}{\Gamma(1+\alpha)}$ and the eigen values (10) are $\lambda_{1,2} = 1 \pm is\sqrt{\frac{k}{m}}$. Now choosing the parameter values $\alpha = 0.8; h = 0.01; m = 1; k = 0.2$ with the initial conditions $x(0) = 0.5; y(0) = 0.5$, we get the eigen values $\lambda_{1,2} = 1 \pm i0.0121$. The system is undamped, the solution has led to oscillations that become unbounded, refer figure - 10.

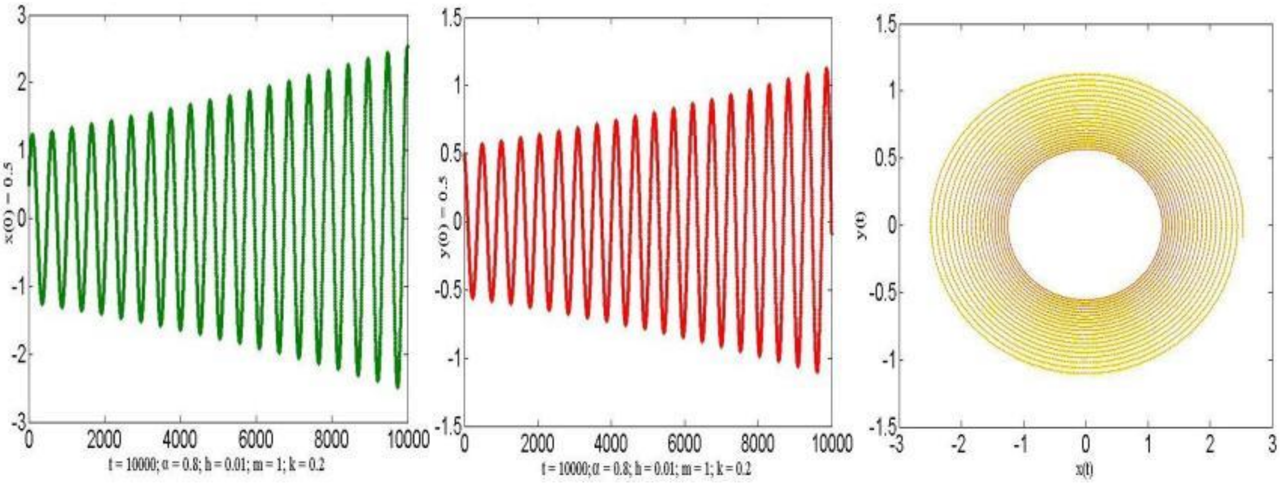


FIGURE 10. Discrete Fractional Order Mass Spring system (10) of Free Damped Motions

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