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A Six-Order Method for Non-linear Equations to Find Roots

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Abstract- *In this paper, new variant of Newton's method based on harmonic mean has been discussed and its sixth order convergence has been established. The method generates a sequence converging to the root with a suitable choice of initial approximation x_0. In terms of computational cost, it requires evaluations of only two functions and two first order derivatives per iteration and the efficiency index of the proposed method is 1.5651. Proposed method has been compared with some existing methods. Proposed method is free from the evaluation of the second order derivative of the given function as required in the family of Chebyshev–Halley type methods. The efficiency of the method is verified on a number of numerical examples.*

Keywords- *Newton's method, Iteration function, Order of convergence, Function evaluations, Efficiency index.*

I. INTRODUCTION

In many branch of science and engineering, the nonlinear and transcendental problems of the form $f(x) = 0$, are complex in nature. Since the exact solution of the problems are not always possible by the usual algebraic process, therefore numerical iterative methods such as Newton, secant methods are often used to obtain the approximate solution of such problems. Though these methods are very effective, but there are some limitations that they do not give the result as fast as required and takes several iterations or some time methods fails. There are so many methods developed on the improvement of quadratically convergent Newton's method so as to get a better convergence order than Newton. This paper is concerned with the iterative methods for finding a simple root α, i.e. $f(\alpha) = 0$, and $f'(\alpha) \neq 0$ of $f(x) = 0$, where $f : R \to R$, be the continuously differentiable real function.

Now we consider the problem of finding a real zero of a function $f: I \subset R \to R$. It can be determined as a fixed point of some iteration function g by means of the one-point iteration method $x_{n+1} = g(x_n)$, $n = 0, 1, \cdots$, where x_0 is the starting value, The best known and the most widely used example of these types of methods is the classical Newton's method given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad n = 0, 1, \cdots \tag{1}$$

It converges quadratically to simple zeros and linearly to multiple zeros. In the literature, some of its modifications have been introduced in order to accelerate it or to get a method with a higher order of convergence at the expense of additional evaluations of functions, derivatives and changes in the points of iterations. If we consider the definition of *efficiency index* as $p^{1/m}$ where p is the order of the method and m is the number of functions evaluations required by the method (units of work per iteration), then the efficiency index of this method is 1.414. All these modifications are in the direction of increasing the local order of convergence to increase the efficiency index. The method developed by Weerakoon et. al. [1], called as trapezoidal Newton's method or arithmetic mean Newton's method, suggests for some other variants of Newton's method. Frontini et. al. [9] developed new modifications of Newton's method to produce iterative methods with third order of convergence and efficiency index 1.442. With the same efficiency index, Ozban [2], and Traub [23] developed a third order method requiring one function and two first derivatives evaluations per iteration. Chen [10] described some new iterative formulae having third order convergence. Ostrowski [7] developed both third and fourth order methods each requiring evaluations of two functions and one derivative per iteration. Neta [19] developed a family of sixth order methods which requires evaluations of three functions and one first derivative per iteration. Sharma et. al. [22] developed a one parameter family of sixth order methods based on Ostrowski fourth order multipoint method. Each family required three evaluations of the given function and one evaluation of the derivative per iteration. Chun [6] presented a one parameter family of variants of Jarratt's fourth order method for solving nonlinear equations. It is shown there that the order of convergence of each family member is improved from four to six even though it adds one evaluation of the function at the point iterated by Jarratt's method per iteration. Kou et al. [11–13] presented a family of new variants of Chebyshev–Halley methods and also an improvement of Jarratt method. These new methods have sixth order of convergence although they only add one evaluation of the function at the point iterated by Chebyshev–Halley method and Jarratt method. Parhi et. al. [20] developed a sixth order method for nonlinear equations, by extending a third order method of Weerakoon et. al. [1] requires evaluations of two functions and two first derivatives per iteration.

II. DEFINITIONS

Definition 1: If the sequence $\{x_n / n \geq 0\}$ tends to a limit α in such a way that

$$\lim_{x_n \to \alpha} \frac{x_{n+1} - \alpha}{(x_n - \alpha)^p} = C \quad (2)$$

for some $C \neq 0$ and $p \geq 1$, then the order of convergence of the sequence is said to be p, and C is known as *asymptotic error constant*.

When $p = 1$, $p = 2$ or $p = 3$, the sequence is said to convergence *lineally*, *quadratically* and *cubically respectively*. The value of p is called the *order of convergence* of the method which produces the sequence $\{x_n : n \geq 0\}$. Let $e_n = x_n - \alpha$ then the relation $e_{n+1} = C{e_n}^p + O({e_n}^{p+1})$ is called the error equation for the method, p being the order of convergence.

Definition 2: *Efficiency index* is simply defined as $p^{1/m}$ where p is the order of the method and m is the number of functions evaluations required by the method (units of work per iteration).

Therefore, the efficiency index of Newton´s method is 1.414 and iterative methods with order of convergence three has efficiency index 1.442.

III. DESCRIPTION OF THE METHODS

Let α be a simple zero of a sufficiently differentiable function f and consider the numerical solution of the equation $f(x) = 0$, then

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)\,dt. \quad (3)$$

Approximating f' by $f'(x_n)$ on the interval $[x_n, x]$, we get the value $(x - x_n)f'(x_n)$ for the integral in (3) and then putting $x = \alpha$, we get, $0 \approx f(x_n) + (\alpha - x_n)f'(x_n)$, therefore, an approximation for α, known as Newton's method, is given by $x_{n+1} = x_n - f(x_n)/f'(x_n)$, $n = 0, 1, \cdots$. On the other hand, if we approximate the integral in (3) by the trapezoidal rule and then putting $x = \alpha$, we obtain, $0 \approx f(x_n) + (1/2)(\alpha - x_n)(f'(x_n) + f'(\alpha))$. Therefore, an approximation x_{n+1} for α is given by

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x_{n+1})}.$$

Approximating the $(n+1)^{th}$ value by the Newton's method on the right-hand side of the above equation, we have,

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(z_{n+1})}, \quad \text{where } z_{n+1} = x_n - f(x_n)/f'(x_n) \quad (4)$$

for $n = 0, 1, 2, \cdots$, the trapezoidal Newton's method of Weerakoon et. al. [1]. Rewriting equation (4) as

$$x_{n+1} = x_n - \frac{f(x_n)}{(f'(x_n) + f'(z_{n+1}))/2}, \quad n = 0, 1, \cdots,$$

and further we rewrite as:

$$z_n = x_n - \frac{f(x_n)}{(f'(x_n) + f'(y_n))/2}, \quad n = 0, 1, \cdots, \quad (5)$$

So, this variant of Newton's method can be viewed as obtained by using arithmetic mean of $f'(x_n)$ and $f'(y_n)$ instead of $f'(x_n)$ in Newtoṅs method defined by (1) which was called as *arithmetic mean Newton's method.*

3.1. New Variant of Newton's Method

In (5) if we use the harmonic mean instead of arithmetic mean we get

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad (6)$$

$$z_n = x_n - \frac{1}{2} f(x_n) \left\{ \frac{1}{f'(x_n)} + \frac{1}{f'(y_n)} \right\}. \quad (7)$$

Again

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}, \quad n = 0,1,2,\cdots \tag{8}$$

Now using the linear interpolation on two points $(x_n, f'(x_n))$ and $(y_n, f'(y_n))$, we get,

$$f'(x) \approx \frac{x - x_n}{y_n - x_n} f'(y_n) + \frac{x - y_n}{x_n - y_n} f'(x_n) \tag{9}$$

Thus, we approximate $f'(z_n)$ as: $f'(z_n) \approx \frac{z_n - x_n}{y_n - x_n} f'(y_n) + \frac{z_n - y_n}{x_n - y_n} f'(x_n)$, then

$$f'(z_n) \approx \frac{1}{2f'(y_n)} [2f'(x_n)f'(y_n) - \{f'(x_n)\}^2 + \{f'(y_n)\}^2]. \tag{10}$$

Therefore the proposed method can be written as follows:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad z_n = x_n - \frac{1}{2} f(x_n)\left\{\frac{1}{f'(x_n)} + \frac{1}{f'(y_n)}\right\}, \tag{11}$$

$$x_{n+1} = z_n - \frac{2f(z_n)f'(y_n)}{2f'(x_n)f'(y_n) - \{f'(x_n)\}^2 + \{f'(y_n)\}^2}. \tag{12}$$

Clearly this method requires evaluations of only two functions *f* and two derivative *f´* and no second order derivative of *f*.

IV. CONVERGENCE ANALYSIS

Theorem* 1:* Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f: I \subset R \to R$ for an open interval I. If x_0 is sufficiently close to α, then the methods defined by (12) has six order convergence.
Proof. Since $\alpha \in I$ is a simple zero of f, then, we have,

$$f(x_n) = f'(\alpha)\,[\, e_n + C_2 e_n^2 + C_3 e_n^3 + O(e_n^4)], \text{ where } C_j = \left(\frac{1}{j!}\right) f^{(j)}(\alpha)/f'(\alpha), \tag{13}$$

$$f'(x_n) = f'(\alpha)\,[1 + 2C_2\, e_n + 3C_3 e_n^2 + 4C_4\, e_n^3 + O(e_n^4)]. \tag{14}$$

Now from equations (13), (14), we have, $f(x_n)/f'(x_n) = e_n - C_2 e_n^2 + (2C_2^2 - 2C_3)e_n^3 + O(e_n^4)$. Therefore, from the first equation of (11), we get, $y_n = \alpha + C_2 e_n^2 + (2C_3 - 2C_2^2)e_n^3 + O(e_n^4)$. Now expanding $f'(y_n)$ about α, we get,

$$f'(y_n) = f'(\alpha)[1 + 2C_2^2 e_n^2 + 4C_2(C_3 - C_2^2)e_n^3 + O(e_n^4)]. \tag{15}$$

Substituting the value of equations (13), (14) and (15) in second equation of (11), we get,

$$z_n = \alpha + e_n - \frac{1}{2}\{e_n + C_2 e_n^2 + C_3 e_n^3 + O(e_n^4)\}\left[\frac{1}{1 + 2C^2 e_n + 3C^3 e_n^2 + 4C^4 e_n^3 + O(e_n^4)} + \frac{1}{1 + 2C_2^2 e_n^2 + 4C_2(C_3 - C_2^2)e_n^3 + O(e_n^4)}\right]$$

$$\Rightarrow z_n = \alpha + \frac{1}{2} C_3 e_n^3 + O(e_n^4).$$

Again

$$f'(x_n)f'(y_n) = \{f'(\alpha)\}^2[1 + 2C_2 e_n + (2C_2^2 + 3C_3)e_n^2 + 4(C_2C_3 + C_4)e_n^3 + O(e_n^4)],$$

$$\{f'(x_n)\}^2 = \{f'(\alpha)\}^2[1 + 4C_2 e_n + (4C_2^2 + 6C_3)e_n^2 + (12C_2C_3 + 8C_4)e_n^3 + O(e_n^4)],$$

$$\{f'(y_n)\}^2 = \{f'(\alpha)\}^2[1 + 4C_2^2 e_n^2 + (8C_2C_3 - 8C_2^3)e_n^3 + O(e_n^4)].$$

Therefore from equation (10), we have, $f'(z_n) \approx f'(\alpha)[1 - 2C_2C_3 e_n^3 + O(e_n^4)]$. Hence from $x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}$, we get,

$$x_{n+1} = \alpha + \frac{1}{2} C_3 e_n^3 + O(e_n^4) - f\left(\alpha + \frac{1}{2} C_3 e_n^3 + O(e_n^4)\right)\frac{1}{f'(\alpha)}[1 - 2C_2C_3 e_n^3 + O(e_n^4)]^{-1}.$$

Using Taylor's expansion and the fact that $f(\alpha) = 0$, we get,

$$e_{n+1} = -\frac{3}{4} C_2 C_3^2 e_n^6 + O(e_n^7).$$

This shows that method defined by (12) has sixth order convergence.
Remark: If second order derivative or third order derivative is be zero at the root then the order of convergence increases up to seven or eight.

V. NUMERICAL RESULTS AND CONCLUSION

In this section, we presented the results of some numerical tests to compare the efficiency of the proposed method. In Table 1, we give the number of iterations (N) and the number of function evaluations (NOFE) required satisfying the stopping criterion. PM denotes proposed method. Kou and Li is the method [13], an improvement of Jarratt method. Gupta denotes for [20] Parhi and Gupta, A sixth order method for nonlinear equations. Numerical computations have been carried out in MATLAB. The stopping criterion has been taken as $|x_{n+1} - \alpha| + |f(x_{n+1})| < 10^{-14}$. In Table 1 for simple roots following test functions have been used.

Table 1 - Comparison with existing sixth order methods

F(x)	x_0	N			NOFE		
		Kou and Li	Gupta	PM	Kou and Li	Gupta	PM
(a)	-0.7	17	33	4	68	132	16
	1	2	2	2	8	8	8
	3	2	3	2	8	12	8
(b)	0.3	4	4	4	16	16	12
	2	2	2	2	8	8	8
	3	2	2	2	8	8	8
(c)	0.5	4	5	4	16	20	16
	2	2	2	2	8	8	8
	3	2	2	2	8	8	8
(d)	0.3	2	3	2	8	12	8
	1	2	2	2	8	8	8
(e)	0.3	2	2	2	8	8	8
	0.5	2	2	2	8	8	8
	0.7	3	2	2	12	8	8
(f)	0.1	5	4	4	20	16	16
	2.5	2	2	2	8	8	8
	3.5	3	3	3	12	12	12
(g)	0.8	3	3	3	12	12	12
	1.5	3	4	3	12	16	12
(h)	2	2	2	2	8	8	8
	3	2	2	2	8	8	8
	4	3	3	3	12	12	12
(i)	1	180	9	57	720	36	228
	3.5	5	6	5	20	24	20
	4.5	8	11	9	320	44	36

(a) $x^3 + 4x^2 - 10$, $\alpha = 1.365230013414097$,
(b) $\sin^2 x - x^2 + 1$, $\alpha = -1.404491648215341$,
(c) $x^3 - 10$ $\alpha = 2.154434690031884$,
(d) $x^3 - e^{-x}$ $\alpha = 0.772882959149210$,
(e) $x \sin(1/x) - 0.2\, e^{-x}$, $\alpha = 0.363715708657122$,
(f) $(x - 1)^3 - 1$, $\alpha = 2$,
(g) $x^{10} - 1$, $\alpha = 1$
(h) $x^3 - e^{-x} - 3x + 2$, $\alpha = 0.257530285439861$
(i) $(x - 2)^{23} - 1$, $\alpha = 3$.

Thus the proposed sixth order method for finding simple real roots of nonlinear equations, is free from second order derivative of the given function, as required in the family of Chebyshev–Halley type methods. Method proposed in this paper requires evaluations of two functions and evaluations two first order derivatives per iteration. The convergence analysis of the method is performed to show that the order of convergence of the method is six. The high order convergence is also corroborated by numerical tests

Method has the efficiency index equal to 1.5651, which is better to Newton's method with efficiency index equal to 1.414 and the classical third order methods (1.442), such as Weerakoon and Fernando method, Chebyshev's method, Halley's method and Super-Halley method, fifth order method (1.495) of Kou, Li, Wang [24]. When compared with the sixth order methods of Parhi, Gupta, [20] and Kou and Li [13], proposed method behaves either similarly or better on the examples considered.

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