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Numerical Solutions of Second Order Boundary Value Problems by Hermite Polynomial Methods

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Abstract — *In this paper, we solve numerically second order linear boundary value problems by the technique of Hermite Polynomial methods. For this, we derive a simple and efficient matrix formulation using Hermite polynomials. The proposed method is tested on several numerical examples of second order linear boundary value problems with Neumann and Cauchy types boundary conditions. The approximate solutions of some examples coincide with the exact solutions on using a very few Hermite polynomials. The approximate results, obtained by the propose method, confirm the convergence of numerical solutions and are compared with the existing methods available in the literature.*

Keywords*- Second order boundary value problems, Cauchy boundary condition, Neumann boundary condition***,** *Hermite polynomials, Probabilistic Hermite polynomial.*

I.INTRODUCTION

In science and engineering, there are many linear and nonlinear problems of second order differential equations with various types of boundary conditions, are solved either analytically or numerically. Numerical simulation in engineering science and in applied mathematics has become a powerful tool to study the physical phenomena, particularly when analytical solutions are not available. For example the diffusion occurring in the presence of exothermic chemical reaction, heat conductions associated with radiation effect [6]. Solving such type of boundary value problems analytically is possible only in very rare cases. Study in this field is very interesting. Various methods are available in the literature concerning their numerical solutions [11-12]. Khan [11] obtained a parametric cubic spline solution of two point boundary value problems, Feng and Li [14] solved a second-order Neumann boundary value problem with singular nonlinearity for exact three positive solutions, Lima and Carpentier [10] obtained a Numerical solution of a singular boundary-value problem in non-Newtonian fluid mechanics, Rashidinia and Jalilian[8] introduced a spline solution of two point boundary value problems, Viswanadham et al.[7] obtained a numerical solution of a fourth order boundary value problems by Galerkin method with Quintic B-splines basis, Das et al.[13] produced a method for solutions of nonlinear second order multi-point boundary value problems and recently Bhatti and Bracken[8] solved linear and non-linear differential equation numerically by Galerkin method with Bernstein polynomials basis.

However, in this paper a very simple and efficient matrix is obtained by using Hermite polynomials. The formulation is derived to solve second order boundary value problem with two different cases of boundary conditions, in details, in Section 3. In Section 2, we give a short introduction of Hermite polynomials. Finally, one example of Neumann boundary value problems and two examples of Cauchy boundary value problems are given to verify the proposed. formulation. The results of each example indicate the convergence numerical solutions. Moreover, this method can provide the exact solutions, even with a few lower order Hermite polynomials, if the equation is simple.

II.HERMITE POLYNOMIALS

The general form of the Hermite polynomials of n^{th} degree is defined by

$$H_n(x)=(-1)^n e^{x^2/2}\frac{d^n}{dx^n}\left(e^{-x^2/2}\right) \qquad \text{where n = 0, 1, 2, 3.......} \tag{2.1}$$

The Hermite Polynomials up to tenth degree are as follows:

$$H_0(x)=1$$
$$H_1(x)=x$$

$$H_2(x)=x^2-1$$
$$H_3(x)=x^3-3x$$
$$H_4(x)=x^4-6x^2+3$$
$$H_5(x)=x^5-10x^3+15x$$
$$H_6(x)=x^6-15x^4+45x^2-15$$
$$H_7(x)=x^7-21x^5+105x^3-105x$$
$$H_8(x)=x^8-28x^6+210x^4-420x^2+105$$
$$H_9(x)=x^9-36x^7+378x^5-1260x^3+945x$$
$$H_{10}(x)=x^{10}-45x^6+630x^6-3150x^4+4725x^2-945$$

Now, the Hermite Polynomials which are orthogonal in [-∞, ∞] with respect to the weight function w(t) $e^{-x^2/2}$ can be determined with the aid of the following recurrence formula

$$H_0(t)=1$$
$$H_1(t)=t$$
$$H_{m+1}(t)=\left(\frac{2m+1}{m+1}\right)t\,H_m(t)-\left(\frac{m}{m+1}\right)H_{m-1}(t),\quad m=2,3,4$$

These functions are orthogonal in sub-interval [a, b] with the aid of the following linear transformation

$$\tau=\left(\frac{2}{b-a}\right)t-\left(\frac{b+a}{b-a}\right) \tag{2.2}$$

We have

$$H_0(t)=1$$
$$H_0(t)=\left(\frac{2}{t_f}\right)t-1$$
$$H_m(t)=H_m\left(\frac{2}{t_f}(t)-1\right),$$

These are named shifted Hermite Polynomials in the interval [0,t],[0, t_f]

III. FORMULATION OF MATRIX FORM

Let us consider

$$y''(t)+p(t)y'(t)+q(t)y(t)=r(t),a\le t\le b \text{ , where } y(a)=k_1, y(b)=k_2 \tag{3.1}$$

Where $p(t), q(t) r(t),$ are continuous functions of t on $[a, b]$, k_1 and k_2 are known constants.

Let $$y(t)=\sum_{i=0}^{M+2}C_iH_i(t) \tag{3.2}$$

be an approximate solution of the given BVP (3.1) where M is the total numbers of subinterval of [a,b].

So putting $y(t)=\sum_{i=0}^{M+2}C_iH_i(t)$ in (3.1) we get

$$\sum_{i=0}^{M+2}C_iH''_i(t)+p(t)\sum_{i=0}^{M+2}C_iH'_i(t)+q(t)\sum_{i=0}^{M+2}C_iH_i(t)=r(t) \tag{3.3}$$

Let $a = t_0 < t_1 < t_2 < .... < t_m = b$ be the grid points then at $t = t_i\left(j = 12,3,.....M\right)$ the equation (3.3) becomes

$$\sum_{i=0}^{M+2} C_i \left[H_i^{''}(t_j) + p(t_j)H_i^{'}(t_i) + q(t_j)H_i(t_j)\right] = r(t_j),\ j = 0,1,2,3,....M \tag{3.4}$$

Now for M=5 from the above equation (3.4) for j = 0, 1… 5 and conditions from equation (3.1) we get following matrix system.

We have $[K]_{8*8}\{C\}_{8*1} = [F]_{8*1}$

So $\{C\}_{8*1} = [K]^{-1}{}_{8*8}[F]_{8*1}$

Where $[K]_{8*8}$ =

$$\begin{bmatrix} H_0(a) & H_1(a) & .... & H_6(a) & H_7(a) \\ H''_0(t_0) + p(t_0)H_0'(t_0) + q(t_0)H_0(t_0) & .... & .... & .... & H''_7(t_0) + p(t_0)H_7'(t_0) + q(t_0)H_7(t_0) \\ H''_0(t_1) + p(t_1)H_0'(t_1) + q(t_1)H_0(t_1) & .... & .... & .... & H''_7(t_1) + p(t_1)H_7'(t_1) + q(t_1)H_7(t_1) \\ H''_0(t_2) + p(t_2)H_0'(t_2) + q(t_2)H_0(t_2) & .... & .... & .... & H''_7(t_2) + p(t_2)H_7'(t_2) + q(t_2)H_7(t_2) \\ H''_0(t_3) + p(t_3)H_0'(t_3) + q(t_3)H_0(t_3). & .... & .... & .... & H''_7(t_3) + p(t_3)H_7'(t_3) + q(t_3)H_7(t_3) \\ H''_0(t_4) + p(t_4)H_0'(t_4) + q(t_4)H_0(t_4). & .... & .... & .... & H''_7(t_4) + p(t_4)H_7'(t_4) + q(t_4)H_7(t_4) \\ H''_0(t_5) + p(t_5)H_0'(t_5) + q(t_5)H_0(t_5) & .... & .... & .... & H''_7(t_5) + p(t_5)H_7'(t_5) + q(t_5)H_7(t_5) \\ H_0(b) & H_1(b) & .... & H_6(b) & H_7(b) \end{bmatrix}$$

&

$$[F]_{8*1} = [k_1, r(t_0), r(t_1), r(t_2), r(t_3), r(t_4), r(t_5), k_2]^T$$

Solution of above system gives the unknowns Cj, j = 0, 1, …., M+2 which are used in equation (3.2) to get approximate solution of the BVP (3.1).

3.1. Numerical examples

In this section, we consider two examples of Cauchy boundary value problem and one example of Neumann boundary value problem which are available in the literature. For each example we find the approximate solution using same number of piecewise Hermite polynomials.

3.1.1. Cauchy boundary value problem

Consider the following BVP

$$y'' + y = 3x^2,\ y(0) = 0,\ y(2) = 3{,}5;\ [3] \tag{3.1.1.1}$$

We apply the above method with M = 4 and n = 6 and using the linear transformation (2.2) for converting equation (3.1.1.1) to [0, 1] and we get the approximate solution which is compare with the exact solution given by the following table.

Table 3.1.1.1 . Comparison of y (t) with exact solution.

t	Exact Solution	Approximate Solution by Present Method
$t_0 = 0.0$	0.00	0.0000
$t_1 = 0.5$	0.20	0.1990
$t_2 = 1.0$	0.24	0.2390
$t_3 = 1.5$	1.17	1.1709
$t_4 = 2.0$	3.50	3.4999

3.1.2. . Cauchy boundary value problem

Consider the following BVP

$$-y'' + y = 0,\ y(0) = 0,\ y(2) = 200$$

Using the above method with M = 4 and n = 6, we get the approximate solutions which are compared with the exact solutions as in table.

Table 3.1.2.1. Comparison of y (t) with exact solution.

t	Exact Solution	Approximate Solution by Present Method
$t_0 = 0.0$	0.0000	0.0000
$t_1 = 0.5$	28.7353	28.7364
$t_2 = 1.0$	64.8054	64.8053
$t_3 = 1.5$	117.4171	117.4157
$t_4 = 2.0$	200.0000	199.9999

3.1.3. Neumann boundary value problem

Consider the following BVP

$u''+u=-1,\ 0<x<1$

$$\text{where } u'(0)=\frac{1-\cos 1}{\sin 1}, u'=-\frac{1-\cos 1}{\sin 1}$$

with M = 5 and n =7 and we obtain the approximate solution of the problem as shown in the table.

Table 3.1.3.1. Comparison of u (t) with exact solution.

x	Exact Solution	Approximate Solution by HPM
0.2	0.08860	0.08863
0.4	0.13380	0.13567
0.6	0.13380	0.13380
0.8	0.08860	0.08839
1.0	0.00000	0.00006

IV. CONCLUSION

In this paper, we have developed Hermite Polynomial method to approximate the solution of second order boundary value problems with Neumann and Cauchy boundary conditions. It is observed that the approximate results converge monotonically to the exact solutions. We may realize that this method may be applied to solve other higher order linear boundary value problems for the desired accuracy. The objective of this paper is to present a simple and accurate method to solve second order boundary value problems

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