Scientific Journal of Impact Factor (SJIF): 4.72

e-ISSN (O): 2348-4470
p-ISSN (P): 2348-6406

International Journal of Advance Engineering and Research Development

Volume 4, Issue 4, April -2017

α - graceful labeling and its application to produce graceful related families

V. J. Kaneria[1], Kalpesh M. Patadiya[2]

[1](Department of Mathematics, Saurashtra University, Rajkot, India)
[2](School of Engineering, RK University, Rajkot, India)

Abstract:
In this paper we focused to produce graceful labeling, odd graceful labeling, k - graceful labeling for the graph $G_1 \cup G_2$, among G_1 is α - graceful graph and G_2 has one of above three labeling.

Keywords:
Graceful labeling, α - graceful graph, odd graceful labeling, k - graceful labeling.
AMS Subject Classification number: 05C78.

1 Introduction

In 1966 Rosa[1] defined a function f is a graceful labeling of a graph G, if $f: V(G) \to \{0,1, \dots |E(G)|\}$ is injective and its edge induced function $f^*: E(G) \to \{1,2, \dots, |E(G)|\}$ defines as $f^*(e) = |f(u) - f(v)|$ is bijective, for every edge e = (u,v) ∈ E(G). A graph G is called graceful graph, if it admits a graceful labeling. He also defined α - labeling as a graceful labeling with an additional property that ∃ a non - negative integer $k (0 \leq k < |E(G)|)$ ∋ for every edge e = (u,v) ∈ E(G) , either f(x) ≤ k < f(y) or f(y) ≤ k < f(x) . A graph which admits an α - labeling is must be a bipartite graph and we call here an α - graceful graph.

A function f is called a (k,d) - graceful labeling of a graph G, if $f: V(G) \to \{0,1, \dots, k + (q-1)d\}$ is injective and its edge induced function f^* on E(G), gives edge label absolute difference of its end vertices under f and it produces precisely k, k+d, k+2d,..., k+(q-1)d edge labels, where q = |E(G)| such graph G is called a (k,d) - graceful graph. (k,1) - graceful graph is known as k - graceful graph and (1,2) - graceful graph is known as odd graceful graph. It is obvious that (1,1) - graceful or 1 - graceful labeling is precisely graceful labeling for a graph G.

In [3,4] Kaneria and Jariya defined a semi smooth graceful labeling for a bipartite graph G, it is an injective map $f: V(G) \to \{0,1, \dots, t-1, t+l, t+l+1, \dots, q+l\}$ and its edge induced function $f^*: E(G) \to \{1+l, 2+l, \dots, q+l\}$ defined as $f^*(e) = |f(u) - f(v)|$ is bijective, for each edge e = (u,v) ∈ E(G), for any non - negative integer l. Kaneria, Meera Meghpara and Maulik Khoda[5] proved that semi smooth graceful labeling for a graph G is equivalent to an α - graceful labeling for the graph G.

Here we considered G = (V,E), a simple, finite and undirected graph with p vertices and q edges. We follow Harary[2] for the basic notation and terminology of graph theory.

2 Main Results

Theorem - 2.1

Let G_1 be an α - graceful graph with an α - labeling $f: V(G_1) \to \{0,1, \dots, q_1\}$, where $q_1 = |E(G_1)|$ and a non negative integer k ($0 \leq k < q_1$), for each e = (x, y) $\in E(G_1)$, it satisfies min { f(x), f(y)} ≤ k < max {f(x), f(y)}. Let G_2 be an graceful graph with a graceful labeling $g: V(G_2) \to \{0,1, \dots, q_2\}$, where $q_2 = |E(G_2)|$ and $|f(V(G_1)) \cap \{k-1, k+2\}| \leq 1, |g(V(G_2)) \cap \{1, q_2 - 1\}| \leq 1$. Then $G_1 \cup G_2$ is a graceful graph.

Proof: Let $V_1 = \{v \in V(G_1) / f(v) \leq k\}$ and $V_2 = V(G_1) - V_1$. Since, $q_2 - g$ is a graceful labeling for G_2 and $|g(V(G_2)) \cap \{1, q_2 - 1\}| \leq 1$, without loss of generality we assume that $q_2 - 1 \notin g(V(G_2))$.

To define $h: V(G_1 \cup G_2) \to \{0,1, \dots, q_1 + q_2\}$, we need following two cases.

Case - I: $k + 2 \notin f(V(G_1))$. By property of f and $k, k+2 \notin f(V_2)$.

Define $h/V_1 = f/V_1, h/V_2 = f/V_2 + q_2$ and $h/v(G_2) = g/V(G_2) + k + 2$. It is obvious that $h(V_1) \subseteq \{1,2, \dots, k\}$,$h(V(G_2)) \subseteq \{k+2, k+3, \dots, q_2 + k, q_2 + k + 2\}$, as $q_2 - 1 \notin g(V(G_2))$.

Thus, above defined labeling pattern give rise h is an injective map in this case, as f and g both are injective maps.

Case - II : $k - 1 \notin f(V(G_1))$ By property of f and $k, k-1 \notin f(V_1)$.

Define $h/V_1 = f/V_1, h/V_2 = f/V_2 + q_2$ and $h/V(G_2) = q_2 - g/V(G_2) + k - 1$. It is obvious that $h(V_1) \subseteq \{1,2, \dots, k-2, k\}$, as $k - 1 \notin f(V_1)$, $h(V(G_2)) \subseteq \{k-1, k+1, k+2, k+3, \dots, q_2 + k - 1\}$, as $q_2 - 1 \notin g(V(G_2))$ and $h(V_2) \subseteq \{q_2 + k + 1, q_2 + k + 2, \dots, q_1 + q_2\}$.

Thus, above defined labeling pattern give rise h is an injective map in this case, as f and g both are injective maps.

By construction of $h, h^*/E(G_1) = q_2 + f^*/E(G_1)$ and $h^*/E(G_2) = g^*/E(G_2)$. Therefore, h^* is a bijection on $G_1 \cup G_2$ and so, h is a graceful labeling for $G_1 \cup G_2$. Thus, $G_1 \cup G_2$ is graceful graph.

Illustration - 2.2

$K_{3,4} \cup (P_3 \times P_3)$ and its graceful labeling, $K_{3,4}$ and its α - graceful labeling, $(P_3 \times P_3)$ and its graceful labeling are shown in Figure 1.

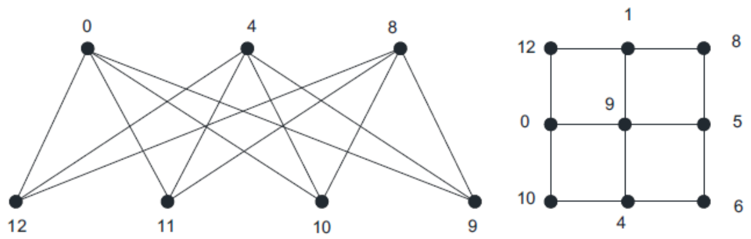

Figure 1:$K_{3,4}$ and its α - graceful labeling[Here k = 8], $(P_3 \times P_3)$ and its graceful labeling. Here $7 \notin f(V(G_1))$ and $11 \notin g(V(G_2))$.

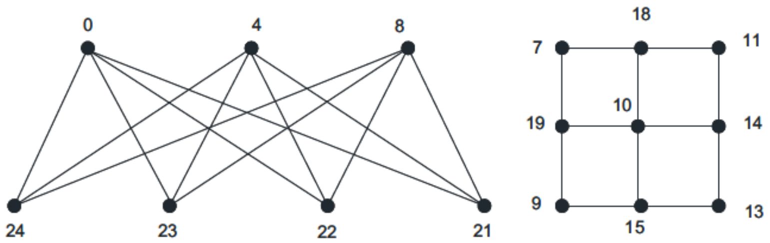

Figure 2: $K_{3,4} \cup (P_3 \times P_3)$ and its graceful labeling.

Theorem - 2.3

Let G_1 be an α - graceful graph and G_2 be an odd graceful graph, then $G_1 \cup G_2$ is also an odd graceful graph.

Proof: Let G_1 be an α - graceful graph with a labeling $f: V(G_1) \to \{0,1,\dots,q_1\}$, where $q_1 = |E(G_1)|$ and a non - negative integer $k (0 \le k \le q_1)$, for each $e = (x, y) \in E(G_1)$, it satisfies the condition $min\{f(x), f(y)\} \le k < max\{f(x), f(y)\}$. Take $V_1 = \{v \in V(G_1)/f(v) \le k\}$ and $V_2 = V(G_1) - V_1$. Let $g: V(G_2) \to \{0,1,\dots,2 \cdot q_2 - 1\}$ be an odd graceful labeling for G_2, where $q_2 = |E(G_2)|$.

Define $h: V(G_1 \cup G_2) \to \{0,1,\dots,2(q_1 + q_2) - 1\}$ as follows.

$h/V_1 = 2(f/V_1)$, $h/V_2 = 2(f/V_2) + 2q_2 - 1$ and $h/V(G_2) = g/V(G_2) + 2k + 1$. It is obvious that $h(V_1) \subseteq \{1,2,\dots,2k\}$, $h(V(G_2)) \subseteq \{2k+1, 2k+2, \dots, 2(q_2 + k)\}$ and $h(V_2) \subseteq \{2(q_2 + k) + 1, 2(q_2 + k) + 2, \dots, 2(q_1 + q_2) - 1\}$. Thus, above defined labeling pattern give rise h is an injective map, as f and g both are injective maps.

By construction of h, $h^*/E(G_1) = 2f^*/E(G_1) = 2f^*/E(G_1) + 2q_2 - 1$ and $h^*/E(G_2) = g^*/E(G_2)$. Therefore, $h^*: E(G_1 \cup G_2) \to \{1,3,\dots,2(q_1 + q_2) - 1\}$ is a bijection and so, $G_1 \cup G_2$ is an odd graceful graph.

Illustration - 2.4

$(P_3 \times P_4 \cup C_6)$ and its odd graceful labeling, $(P_3 \times P_4)$ and its α - labeling, C_6 and its odd graceful labeling are shown in the Figure 4,3.

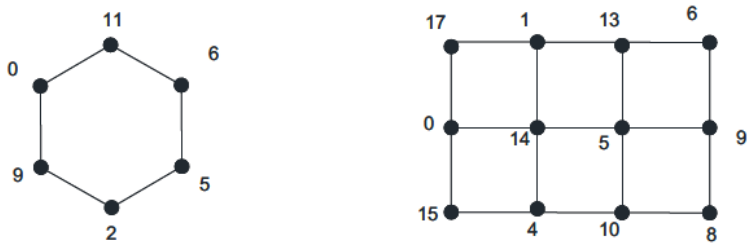

Figure 3: C_6 and its odd graceful labeling, $P_3 \times P_4$ and its α - labeling [Here k = 8].

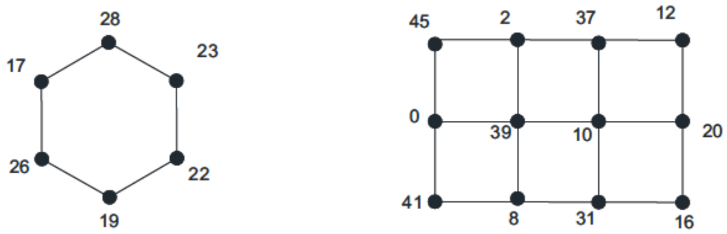

Figure 4: $(P_3 \times P_4 \cup C_6)$ and its odd graceful labeling.

Theorem - 2.5

Let G_1 be an α - graceful graph with an α - labeling $f: V(G_1) \to \{0,1,\dots,q_1\}$, where $q_1 = |E(G_1)|$ and a non - negative integer $t(0 \le t < q_1)$, for each $e = (x,y) \in E(G_1)$, satisfies $min\{f(x), f(y)\} \le t < max\{f(x), f(y)\}$. If G_2 is a k - graceful graph with a k - graceful labeling $g: V(G_2) \to \{0,1,\dots,q_2+k-1\}$, $q_2 = |E(G_2)|, k + q_2 - 2 \notin g(V(G_2))$ and $t + 2 \notin f(V(G_1))$, then $G_1 \cup G_2$ is a k - graceful graph.

Proof : Let $V_1 = \{u \in V(G_1)/f(u) \le t\}$ and $V(G_2) = V(G_1) - V_1$.

Take $V_3 = \{w \in V(G_2)/g(w) < k\}$ and $V_4 = V(G_2) - V_3$.

Define $h: V(G_1 \cup G_2) \to \{0, 1, \dots, q_1 + q_2 + k - 1\}$ as follows.

$h/V_1 = f/V_1$, $h/V_2 = f/V_2 + q_2 + k - 1$ and $h/V(G_2) = g/V(G_2) + t + 2$. By property of f and$t, t + 2 \notin f(V_2)$. It is observed that $h(V_1) \subseteq \{0,1,\dots,t\}, h\big(V(G_2)\big) \subseteq \{t+2, t+3, \dots, t+k+q_2-1, t+k+q_2+1\}$ as $k + q_2 - 2 \notin g(V(G_2))$ and $h(V_2) \subseteq \{k+t+q_2, k+t+q_2+2, \dots k+q_1+q_2-1\}$, as $t + 2 \notin f(V_2)$. Thus, above labeling pattern give rise h is an injective map, as f and g both are injective maps. By construction of $h, h^*/E(G_2) = g^*/E(G_2)$ and $h^*/E(G_1) = f^*/E(G_1) + k + q_1 + q_2 - 1$. Therefore, h^* is a bijection on $G_1 \cup G_2$ and so, h is k - graceful labeling for $G_1 \cup G_2$. Hence, it is a k - graceful graph.

Illustration - 2.6

$G_1 = P_4 \times P_3, G_2 = K_{2,3} \cup P_5$. $G_1 \cup G_2$ and its k - graceful labeling, G_1 and its α - labeling, G_2 and its k - graceful labeling are show in Figure 6,5.

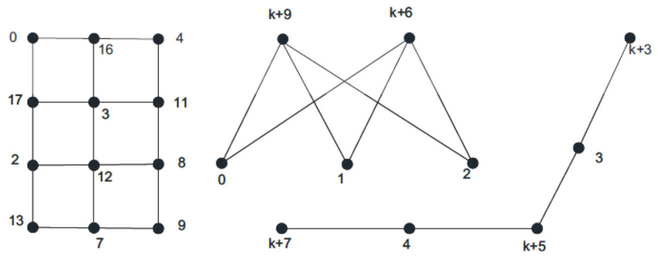

Figure 5: $P_4 \times P_3$ and its α - labeling[here t = 8], $K_{2,3} \cup P_5$ and its k - graceful labeling. where $t + 2 \notin f(V(P_4 \times P_3))$ and

$k + q_2 - 2 = k + 8 \notin f(V(G_2))$

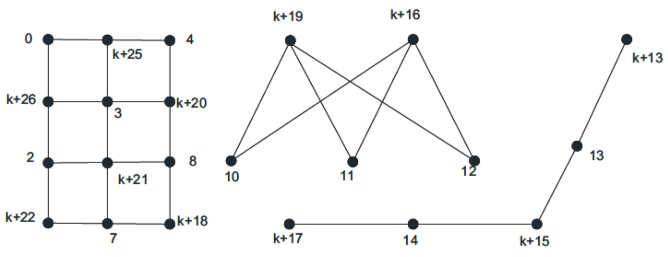

Figure 6: $P_4 \times P_3 \cup (K_{2,3} \cup P_5)$ and its k - graceful labeling.

References:

[1] A. Rosa, On certain valuations of the vertices of a graph, *Theory of Graphs (Internat. Symposium, Rome, July 1966)*, Gordon and Breach, N. Y. and Dunod Paris (1967) pp. 349-355.

[2] F. Harary, Graph Theory, *Addition Wesley, Massachusetts*, (1972).

[3] V. J. Kaneria and M. M. Jariya, Smooth graceful graphs and its application to construct graceful Graphs, *I.J. of Sci. and Research*, 3(8)(2014), 909-912. [http://www.ijsr.net/archive/v3i8/MDIwMTU0ODg=.pdf]

[4] V. J. Kaneria and M. M. Jariya, Semi smooth graceful graph and construction of new graceful trees, *Elixir Appl. Math.*, 76(2014) pp.28,536-28,538.

[5] V.J. Kaneria, Meera Meghpara, and Maulik Khoda, Semi smooth graceful labeling and its application to produce α-labeling, *J. of Graph Labeling*, 2(2), (2016) pp. 153 - 160.

[6] Gallian J.A., A dynamic Survey of graph labeling, *The electronics J. Combin.*,17, # DS6,(2014).