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Solution of higher order ordinary differential equations by using Homotopy analysis method

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Abstract — *In this paper, higher order ordinary differential equations like linear, non linear, homogeneous and non homogeneous are solved by using homotopy analysis method.*

Keywords-*linear, non linear, homotopy analysis method, higher order ordinary differential equation*

I. INTRODUCTION

Higher order boundary value problems arise in the study of fluid dynamics, hydrodynamic, astrophysics, hydro magnetic stability, astronomy, beam and long wave theory, induction motors, engineering and applied physics. The boundary value problems of higher order have been examined due to their mathematical importance and applications in diversified applied sciences

The HAM contains a certain auxiliary parameter h which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution. Moreover, by means of the so-called h -curve, it is easy to determine the valid regions of h to gain a convergent series solution. Thus, through HAM, explicit analytic solutions of non linear problems are possible.

II. HOMOTOPY ANALYSIS METHOD

We consider the following differential equations,

$$N_i\left[S_i\left(x,t\right)\right]=0, i=1,2,\ldots,n$$

Where N_i are nonlinear operators that the represents the whole equations, x and t are independent variables and $S_i(x,t)$ are unknown functions respectively.

By means of generalizing the traditional homotopy method, Liao constructed the so-called zero-order deformation equations

$$(1-q)L\left[\varnothing_i\left(x,t,;q\right)-S_{i,0}\left(x,t\right)\right]=qh_iN_i\left[\varnothing_i\left(x,t,;q\right)\right] \qquad (1)$$

Where $q\in[0,1]$ is an embedding operators, h_i are nonzero auxiliary functions, L is an auxiliary linear operator, $S_{i,0}(x,t)$ are initial guesses of $S_i(x,t)$ and $\varnothing_i\left(x,t,;q\right)$ are unknown functions.

It is important to note that, one has great freedom to choose auxiliary objects such as h_i and L in HAM.

When $q=0$ *and* $q=1$ we get by (1),

$$\varnothing_i\left(x,t,;0\right)=S_{i,0}\left(x,t\right) and \varnothing_i\left(x,t,;1\right)=S_i\left(x,t\right)$$

Thus q increase from 0 to 1, the solutions $\varnothing_i\left(x,t,;q\right)$ varies from initial guesses $S_{i,0}(x,t)$ to $S_i(x,t)$.

Expanding $\varnothing_i\left(x,t,;q\right)$ in Taylor series with respect to ,

$$\varnothing_i\left(x,t,;q\right)=S_{i,0}\left(x,t\right)+\sum_{m=1}^{\infty}S_{i,m}\left(x,t\right).q^m \qquad (2)$$

Where

$$S_{i,m}\left(x,t\right)=\left[\frac{1}{m!}.\frac{\partial^m\varnothing_i\left(x,t,;q\right)}{\partial q^m}\right]_{q=0} \qquad (3)$$

If the auxiliary linear operator, initial guesses, the auxiliary parameter and auxiliary functions are properly chosen than the series equation (2) converges at $q=1$.

$$\varnothing_i\left(x,t,;1\right)=S_{i,0}\left(x,t\right)+\sum_{m=1}^{\infty}S_{i,m}\left(x,t\right). \qquad (4)$$

This must be one of solutions of the original nonlinear equations.
According to (3), the governing equations can be deduced from the zero-order deformation equations (1).
Define the vectors

$$\overrightarrow{S_{i,n}}=\{S_{i,0}\left(x,t\right),S_{i,1}\left(x,t\right),S_{i,2}\left(x,t\right),\ldots,S_{i,n}\left(x,t\right)\}$$

Differentiating (1) m times with respect to the embedding parameter and the setting $q=0$ and finally dividing them by $m!$

We have the so-called m^{th} order deformation equations

$$L\left[S_{i,m}\left(x,t\right)-\chi_m S_{i,m-1}\left(x,t\right)\right]=h_i R_{i,m}(\overrightarrow{S_{i,m-1}}) \qquad (5)$$

Where

$$R_{i,m}\left(S_{i,m-1)}\right)=\left[\frac{1}{\left(m-1\right)!}\cdot\frac{\partial^{m-1}N_i[\varnothing_i\left(x,t,;q\right)}{\partial q^{m-1}}\right]_{q=0} \qquad (6)$$

$$\chi_m=\begin{cases}0,m\le 1\\1,m>1\end{cases}$$

III. HOMOGENEOUS LINEAR ORDINARY DIFFERENTIAL EQUATION

Consider homogeneous non linear differential equation

$$2u_{xx}-2u_x+3u=0 \qquad (7)$$

Subject to the initial condition

$$u(0)=1,\ u'(0)=2 \qquad (8)$$

To solve this system (7) to (8) by HAM, first we choose initial approximation

$$u_0(x)=1+2x$$

And the linear operator

$$L\left(\phi\left(x\,;q\right)\right)=\frac{\partial\phi\left(x\,;q\right)}{\partial x}$$

With the property $L\left(C\right)=0$ where C is integral constant.
We define system of non-linear operator as

$$N\left(\phi\left(x\,;q\right)\right)=2\frac{\partial^2\phi\left(x\,;q\right)}{\partial x^2}-2\frac{\partial\phi\left(x\,;q\right)}{\partial x}+3\phi\left(x\,;q\right) \qquad (9)$$

Using the above definition, we construct the zeroth-order deformation equations

$$\left(1-q\right)\left[\phi\left(x\,;q\right)-S_0\left(x\right)\right]=qhN\left(\phi\left(x\,;q\right)\right) \qquad (10)$$

Obviously, when $q=0$ and $q=1$ we get

$$\phi\left(x\,;0\right)=S_0\left(x\right)=u_0\left(x\right)\text{ and }\phi\left(x\,;1\right)=u(x) \qquad (11)$$

As q increase 0 to 1, ϕ varies from $u_0\left(x\right)$ to $u\left(x\right)$ Expanding $\phi\left(x\,;q\right)$ in Taylor series with respect to q ,

$$\phi\left(x\,;q\right)=S_0\left(x\right)+\sum_{m=1}^{\infty}S_m\left(x\right)\cdot q^m \qquad (12)$$

Where

$$S_m\left(x\right)=\left[\frac{1}{m!}\frac{\partial^m\phi\left(x\,;q\right)}{\partial q^m}\right]_{q=0} \qquad (13)$$

If the auxiliary linear operator, initial guesses, the auxiliary parameter h and auxiliary functions are properly chosen than the series equation (12) converges at $q=1$.

$$\phi(x;1) = S_0(x) + \sum_{m=1}^{\infty} S_m(x)$$

$$\text{i.e. } u(x) = S_0(x) + \sum_{m=1}^{\infty} S_m(x)$$

This must be one of solutions of the original non linear equations as proved by Liao Define the vectors

$$\overrightarrow{S_n} = \left(S_0(x),\ S_1(x),\ S_2(x),\ \ldots..S_n(x)\right) \quad (14)$$

We have the so-called m^{th} order deformation equations

$$L\left[S_m(x) - \chi_m S_m(x)\right] = hR_m \overrightarrow{S_{m-1}} \quad (15)$$

Where

$$R_m \overrightarrow{S_{m-1}} = \left[\frac{1}{(m-1)!} \frac{\partial^{m-1}\phi(x;q)}{\partial q^{m-1}}\right]_{q=0} \quad (16)$$

$$\text{i.e. } R_m \overrightarrow{S_{m-1}} = 2\left(S_{m-1}\right)_{xx} - 2\left(S_{m-1}\right)_x + 3S_{m-1} \quad (17)$$

$$S_m(x) = \chi_m S_{m-1}(x) + h\int_0^x R_m\left(\overrightarrow{S_{m-1}}\right)\, dx + c \quad (18)$$

Now we will calculate

$$S_1(x) = \chi_1 S_0(x) + h\int_0^x R_1\left(\overrightarrow{S_0}\right)\, dx + c \quad (19)$$

Where

$$R_1\left(\overrightarrow{S_0}\right) = 6x - 1$$

So

$$S_1(x) = h\left(3x^2 - x\right)$$

Now The N^{th} order approximation can be expressed by

$$S(x) = S_0(x) + \sum_{m=1}^{N-1} S_m(x) \quad (20)$$

As $N \to \infty$ we get $S(x) \to u(x)$ with some appropriate assumption of h

IV. NON HOMOGENEOUS LINEAR ORDINARY DIFFERENTIAL EQUATION

Consider non homogeneous linear differential equation

$$3u_{xx} - 2u_x + 2u + 8 = 0 \quad (21)$$

Subject to the initial condition

$$u(0) = 0,\ \ u'(0) = 2 \quad (22)$$

To solve this system (21) to (22) by HAM, first we choose initial approximation

$$u_0(x) = 2x$$

And the linear operator

$$L\left(\phi(x;q)\right) = \frac{\partial\phi(x;q)}{\partial x}$$

With the property $L(C) = 0$ where C is integral constant.

We define system of non-linear operator as

$$N\left(\phi(x;q)\right) = 3\frac{\partial^2\phi(x;q)}{\partial x^2} - 2\frac{\partial\phi(x;q)}{\partial x} + 2\phi(x;q) + 8 \quad (23)$$

Using the above definition, we construct the zeroth-order deformation equations

$$(1-q)\left[\phi(x;q)-S_0(x)\right]=qhN\left(\phi(x;q)\right) \tag{24}$$

Obviously, when $q=0$ and $q=1$ we get

$$\phi(x;0)=S_0(x)=u_0(x) \text{ and } \phi(x;1)=u(x) \tag{25}$$

As q increase 0 to 1, ϕ varies from $u_0(x)$ to $u(x)$ Expanding $\phi(x;q)$ in Taylor series with respect to q ,

$$\phi(x;q)=S_0(x)+\sum_{m=1}^{\infty}S_m(x)\cdot q^m \tag{26}$$

Where

$$S_m(x)=\left[\frac{1}{m!}\frac{\partial^m\phi(x;q)}{\partial q^m}\right]_{q=0} \tag{27}$$

If the auxiliary linear operator, initial guesses, the auxiliary parameter h and auxiliary functions are properly chosen than the series equation (26) converges at $q=1$.

$$\phi(x;1)=S_0(x)+\sum_{m=1}^{\infty}S_m(x)$$

$$\text{i.e. } u(x)=S_0(x)+\sum_{m=1}^{\infty}S_m(x)$$

This must be one of solutions of the original non linear equations as proved by Liao Define the vectors

$$\overrightarrow{S_n}=\left(S_0(x),\ S_1(x),\ S_2(x),\ .....S_n(x)\right) \tag{28}$$

We have the so-called m^{th} order deformation equations

$$L\left[S_m(x)-\chi_m S_m(x)\right]=hR_m\overrightarrow{S_{m-1}} \tag{29}$$

Where

$$R_m\overrightarrow{S_{m-1}}=\left[\frac{1}{(m-1)!}\frac{\partial^{m-1}\phi(x;q)}{\partial q^{m-1}}\right]_{q=0} \tag{30}$$

$$\text{i.e. } R_m\overrightarrow{S_{m-1}}=3\left(S_{m-1}\right)_{xx}-2\left(S_{m-1}\right)_x+2S_{m-1}+8 \tag{31}$$

$$S_m(x)=\chi_m S_{m-1}(x)+h\int_0^x R_m\left(\overrightarrow{S_{m-1}}\right)\ dx+c \tag{32}$$

Now we will calculate

$$S_1(x)=\chi_1 S_0(x)+h\int_0^x R_1\left(\overrightarrow{S_0}\right)\ dx+c \tag{33}$$

Where

$$R_1\left(\overrightarrow{S_0}\right)=4x+4$$

So

$$S_1(x)=h\left[2x^2+4x\right]$$

Now The N^{th} order approximation can be expressed by

$$S(x)=S_0(x)+\sum_{m=1}^{N-1}S_m(x) \tag{34}$$

As $N\to\infty$ we get $S(x)\to u(x)$ with some appropriate assumption of h

V. NON HOMOGENEOUS NON LINEAR ORDINARY DIFFERENTIAL EQUATION

Consider non homogeneous non linear differential equation

$$u_{xx}-\left(u_x\right)^2+4u+5=0 \tag{35}$$

Subject to the initial condition

$$u(0)=1,\ u'(0)=0 \tag{36}$$

To solve this system (35) to (36) by HAM, first we choose initial approximation

$$u_0(x)=1+x^2$$

And the linear operator

$$L(\phi(x;q))=\frac{\partial\phi(x;q)}{\partial x}$$

With the property $L(C)=0$ where C is integral constant.

We define system of non-linear operator as

$$N(\phi(x;q))=\frac{\partial^2\phi(x;q)}{\partial x^2}-\left(\frac{\partial\phi(x;q)}{\partial x}\right)^2+4\phi(x;q)+5 \tag{37}$$

Using the above definition, we construct the zeroth-order deformation equations

$$(1-q)\left[\phi(x;q)-S_0(x)\right]=qhN(\phi(x;q)) \tag{38}$$

Obviously, when $q=0$ and $q=1$ we get

$$\phi(x;0)=S_0(x)=u_0(x) \text{ and } \phi(x;1)=u(x) \tag{39}$$

As q increase 0 to 1, ϕ varies from $u_0(x)$ to $u(x)$ Expanding $\phi(x;q)$ in Taylor series with respect to q ,

$$\phi(x;q)=S_0(x)+\sum_{m=1}^{\infty}S_m(x)\cdot q^m \tag{40}$$

Where

$$S_m(x)=\left[\frac{1}{m!}\frac{\partial^m\phi(x;q)}{\partial q^m}\right]_{q=0} \tag{41}$$

If the auxiliary linear operator, initial guesses, the auxiliary parameter h and auxiliary functions are properly chosen than the series equation (40) converges at $q=1$.

$$\phi(x;1)=S_0(x)+\sum_{m=1}^{\infty}S_m(x)$$

$$\text{i.e. } u(x)=S_0(x)+\sum_{m=1}^{\infty}S_m(x)$$

This must be one of solutions of the original non linear equations as proved by Liao Define the vectors

$$\overrightarrow{S_n}=(S_0(x),\ S_1(x),\ S_2(x),\ .....S_n(x)) \tag{42}$$

We have the so-called m^{th} order deformation equations

$$L\left[S_m(x)-\chi_m S_m(x)\right]=hR_m\overrightarrow{S_{m-1}} \tag{43}$$

Where

$$R_m\overrightarrow{S_{m-1}}=\left[\frac{1}{(m-1)!}\frac{\partial^{m-1}\phi(x;q)}{\partial q^{m-1}}\right]_{q=0} \tag{44}$$

$$\text{i.e. } R_m\overrightarrow{S_{m-1}}=(S_{m-1})_{xx}-S_{m-1}(S_{m-1})_x+4S_{m-1}+5 \tag{45}$$

$$S_m(x)=\chi_m S_{m-1}(x)+h\int_0^x R_m(\overrightarrow{S_{m-1}})\ dx+c \tag{46}$$

Now we will calculate

$$S_1(x)=\chi_1 S_0(x)+h\int_0^x R_1(\overrightarrow{S_0})\ dx+c \tag{47}$$

Where

$$R_1(\overrightarrow{S_0})=-x^4+2x^2+10$$

So

$$S_1(x) = h\left[-\frac{x^5}{5} + \frac{2x^3}{3} + 10\right]$$

Now The N^{th} order approximation can be expressed by

$$S(x) = S_0(x) + \sum_{m=1}^{N-1} S_m(x) \tag{48}$$

As $N \to \infty$ we get $S(x) \to u(x)$ with some appropriate assumption of h

VI. HOMOGENEOUS NON LINEAR ORDINARY DIFFERENTIAL EQUATION

Consider homogeneous non linear differential equation

$$u_{xx} - u \cdot (u_x)^2 + 3u = 0 \tag{49}$$

Subject to the initial condition

$$u(0) = 1, \qquad u'(0) = 1 \tag{50}$$

To solve this system (49) to (50) by HAM, first we choose initial approximation

$$u_0(x) = 1 + x - x^2$$

And the linear operator

$$L(\phi(x;q)) = \frac{\partial \phi(x;q)}{\partial x}$$

With the property $L(C) = 0$ where C is integral constant.

We define system of non-linear operator as

$$N(\phi(x;q)) = \frac{\partial^2 \phi(x;q)}{\partial x^2} - \phi(x;q)\left(\frac{\partial \phi(x;q)}{\partial x}\right)^2 + 3\phi(x;q) \tag{51}$$

Using the above definition, we construct the zeroth-order deformation equations

$$(1-q)\left[\phi(x;q) - S_0(x)\right] = qhN(\phi(x;q)) \tag{52}$$

Obviously, when $q = 0$ and $q = 1$ we get

$$\phi(x;0) = S_0(x) = u_0(x) \text{ and } \phi(x;1) = u(x) \tag{53}$$

As q increase 0 to 1, ϕ varies from $u_0(x)$ to $u(x)$ Expanding $\phi(x;q)$ in Taylor series with respect to q ,

$$\phi(x;q) = S_0(x) + \sum_{m=1}^{\infty} S_m(x) \cdot q^m \tag{54}$$

Where

$$S_m(x) = \left[\frac{1}{m!}\frac{\partial^m \phi(x;q)}{\partial q^m}\right]_{q=0} \tag{55}$$

If the auxiliary linear operator, initial guesses, the auxiliary parameter h and auxiliary functions are properly chosen than the series equation (54) converges at $q = 1$.

$$\phi(x;1) = S_0(x) + \sum_{m=1}^{\infty} S_m(x)$$

$$\text{i.e. } u(x) = S_0(x) + \sum_{m=1}^{\infty} S_m(x)$$

This must be one of solutions of the original non linear equations as proved by Liao Define the vectors

$$\overrightarrow{S_n} = \left(S_0(x),\ S_1(x),\ S_2(x),\ .....S_n(x)\right) \tag{56}$$

We have the so-called m^{th} order deformation equations

$$L\left[S_m(x)-\chi_m S_m(x)\right]=hR_m\overrightarrow{S_{m-1}} \tag{57}$$

Where

$$R_m\overrightarrow{S_{m-1}}=\left[\frac{1}{(m-1)!}\frac{\partial^{m-1}\phi(x;q)}{\partial q^{m-1}}\right]_{q=0} \tag{58}$$

$$\text{i.e. } R_m\overrightarrow{S_{m-1}}=(S_{m-1})_{xx}-S_{m-1}(S_{m-1})^2{}_x+3S_{m-1} \tag{59}$$

$$S_m(x)=\chi_m S_{m-1}(x)+h\int_0^x R_m\left(\overrightarrow{S_{m-1}}\right)\ dx+c \tag{60}$$

Now we will calculate

$$S_1(x)=\chi_1 S_0(x)+h\int_0^x R_1\left(\overrightarrow{S_0}\right)\ dx+c \tag{61}$$

Where

$$R_1\left(\overrightarrow{S_0}\right)=8x-2x^2-8x^3+4x^4$$

So

$$S_1(x)=h\left[4x^2-\frac{2}{3}x^3-2x^4+\frac{4}{5}x^5\right]$$

Now The N^{th} order approximation can be expressed by

$$S(x)=S_0(x)+\sum_{m=1}^{N-1}S_m(x) \tag{62}$$

As $N\to\infty$ we get $S(x)\to u(x)$ with some appropriate assumption of h

VII. CONCLUSION

Homotopy analysis method is very useful for solving various types of homogeneous, non homogeneous, linear, non linear ordinary differential equation. Also, the system of non linear equation can be easily solved by homotopy analysis method due to freedom of choosing the parameter h.

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